What Follows
A Logic and Philosophy Text

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The question of logic is: Does the conclusion certainly follow if the premises be true?
- Augustus De Morgan

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Acknowledgments

This book is the product of both an unease I have felt growing slowly over many years of teaching logic in philosophy departments as well as a sudden decision to ignore prudence. My unease derived from the ways in which even excellent students finished my Logic course without a good understanding of why philosophers studied the topic. I wanted to find ways to show my students some of the connections between formal deductive logic and broader philosophical topics. Coming to Hamilton College, I began teaching Philosophy Fridays in Logic, putting aside the technical material on truth tables and derivations and talking about non-standard topics, ones which appeared in only cursory fashion, if at all, in the standard textbooks. Every other Friday, I would assign a philosophy reading relating to the course material and spend a class talking about how logic and philosophy connect: Goodman's *The Problem of Counterfactual Conditionals*; a selection from Aristotle's *De Interpretatione*; Quine's "On What There Is;" Searle's, "Can Computers Think?" Each student wrote a short paper on some topic raised in a Philosophy Friday.

Students responded well to Philosophy Fridays, but the readings I assigned were often too difficult. As in many departments, Logic at Hamilton attracts students from departments across the college. Most of these students were too unfamiliar with philosophy to work comfortably with the material I found most interesting. I received many fine papers and was convinced that my students were leaving the course with a greater awareness of why we philosophers study logic. But, unfortunate numbers of students let me know that they were not enjoying the readings.

My sudden decision to ignore prudence happened during a logic class, Fall 2010, when I mentioned, for a reason I still don't fully understand, that if anyone wanted to spend time writing logic problems for a summer, I would try to find some funding for the project. Then, I would put together my scattered notes and write the textbook of which I dreamed: lots of logic problems paired with a text on the connections between logic and philosophy which is accessible to students with little or no philosophy background. I suppose that, contrary to the evidence of my conscious thoughts, deep inside I don't much like summer vacations. Jess Gutfleish volunteered for the problem-writing task, and the Dean of Faculty's Office at Hamilton College agreed to fund her work, with a Class of 1966 Faculty Development Award.

In the summer of 2011, Jess and I worked side-by-side in the archaeology teaching lab at Hamilton producing *What Follows* for use the following fall. I wrote the text and she worked amazingly, assiduously, and indefatigably writing the exercises. I had difficulty keeping up with her.

I am ineffably grateful to Jess for all of her hard work and the mountain of insidiously difficult (as well as more ordinary) logic problems she devised. Thank you to Margaret Gentry and the Dean of Faculty's office at Hamilton for their support through the Class of 1966 Grant. I am grateful to the Nathan Goodale and Tom Jones for letting us have their lab in which to work. I also owe thanks to the many students who have helped me construct an innovative Logic course. Most importantly, I am grateful to my wife, Emily, and my children, Marina and Izzy, who suffered through a summer that was less fun than it could have been for them so that I could write the logic book I wanted to use.

Preface
§1: Defining 'Logic'

What Follows is a textbook in formal deductive logic and its relation to philosophy. Its focus is a definition of logical consequence for a variety of formal languages. In the book, we examine, regiment, evaluate, and derive conclusions for arguments.

Let's start by trying to characterize what the terms 'logic' and 'argument' refer to. Consider the following claims which someone might use to define those terms.

LA — Logic is the study of argument.
Arguments are what people who study logic study.

Two aspects of the pair of sentences LA are worth noticing. First, they provide a circular definition which makes the characterizations nearly useless. If you do not understand the terms logic' and 'argument', then the sentences in LA are not going to help you, except for showing that the two terms are related.

Second, the circularity of this pair of definitions is a formal result. It can be applied to any of various other purported definitions. The formal property of circularity can be instantiated in many different contexts, using sentences with many different contents, like the pair of sentences at SS or GW.

SS — Sheep are the things that shepherds tend.
Shepherds are things that tend sheep.

GW — Glubs are extreme cases of woozles.
Woozles are ordinary forms of glubs.

In the cases of LA and SS, you might not notice the problem of the formal property of circularity. In GW, the problem is obvious. Without knowing what glubs and woozles are, GW is useless, and its uselessness is a product of its poor form. This textbook is about such formal results.

In contrast to LA, LB is not formally circular.

LB — Logic is the study of argument.
An argument is a set of statements, called premises, intended to establish a specific claim, called the conclusion.

LB explains the meaning of one term, 'logic', by using other ones. If such a definition is to be informative, these other terms should be more familiar. If not, we can continue the process.

CP — To establish a claim is to justify or provide evidence for it.
A 'proposition', or a 'statement', is a declarative sentence that has a truth value.
The truth values are true and false. (In this text we will mainly focus on just two truth values, though there are logics with more. Some interesting logics have three, or infinitely many.)

Pairing LB and CP, we see a characterization of logic as the rules of what follows from what, of which consequences derive from which assumptions. We make inferences all the time: if I buy this book, I won't have enough money for the cup of coffee I wanted; if I make a turn here, I'll end up in Waterville; she must be angry with me because she hasn't returned my email yet. When we think about the consequences of our actions or the reasons some event has occurred, we are using logic. Good logic is thus a precondition for all good reasoning.

When evaluating an argument, we perform two steps. First, we see whether the argument follows from the premises. An argument whose conclusion follows, formally, from its premises is called valid. Chapter 1 is dedicated to constructing a precise notion of validity, of what follows, for propositional logic. Indeed, the notion of validity is the central topic of the book.

Our second step in evaluating an argument is to see whether the premises are true. In a valid deductive argument, if the premises are true, then the conclusion is necessarily true. This result is what makes deductive logic important.

This textbook is dedicated to the first step in the process of evaluating arguments. The second step is not logical, but scientific. Roughly speaking, we examine our logic to see if our reasoning is acceptable. We examine the world to see if our premises are true. While we prefer our arguments both to be valid and to have true premises, this book is mainly dedicated to the form of the argument, not to its content.

You might wonder whether the logic in this book, formal deductive logic, is descriptive, representing how we actually reason, or prescriptive, setting out rules for proper reasoning. Before we can start to answer this question, we have to see what our logic looks like. The nature of some elementary systems of formal logic is the focus of the first three chapters of this book. In the fourth and last chapter, I discuss a variety of philosophical questions arising from or informed by the study of formal logic. I intend the sections of Chapter 4 to be read along with the formal material in the first three chapters. Each section of Chapter 4 presupposes the understanding of a different amount of formal work and I provide a guide showing when the different sections of Chapter 4 are appropriate to read.

Logic and Languages

There are (at least) three kinds of languages in this book. First, there is the natural language, English, in which most of the book is written. Other natural languages include Spanish and Swahili. Second, there are the formal languages which are the main focus of the first three chapters. I will specify the formal languages precisely.

Between formal and natural languages is a third kind of language made of elements of the first two and used to study a formal language. This metalanguage is mostly English. You might not even think of it as a language separate from English, and for the most part you need not think about the metalanguage too carefully. But it includes some technical terms not in ordinary English. For example, the rules of inference we will examine in Chapter 2 are not really in English; they are written using Greek letters. They are parts of the metalanguage we use to tell us how to work in the object language. We can add these same meta-linguistic rules to any natural language to form a metalanguage. Our metalanguage thus differs from any particular natural language. I will not specify the metalanguage as precisely as the object languages.

It is customary to give names to object languages. Chapters 1 and 2 focus on one object language which I will call **PL**, for propositional logic. Chapter 3 discusses four further formal languages:

M	Monadic (First-Order) Predicate logic
F	Full (First-Order) Predicate logic
FF	Full (First-Order) Predicate logic with Functors
S	Second-Order Predicate logic

In addition to naming each language, I specify formation rules for formulas of the language. I also introduce deductive systems using each language. A deductive system may use more than one language and a language may use more than one deductive system.

A formal object language may be seen as a meaningless tool. In order to apply this tool to actual arguments and inferences, we have to interpret the symbols of the language. No language can determine its own interpretation. So we specify interpretations of object languages by stepping outside of those languages and into metalanguages.

Many students, when they begin to study logic, find it to be like an amusing toy. There are rules for working in the object language, and once you learn those rules, it can be fun to play with them. When I started studying logic, in college, I couldn't believe that one could earn credit for filling out truth tables, translating English into formal languages, and constructing derivations. To me, a person who loves puzzles and games, and who has a very modest capacity for mathematics, logic was too much fun to be serious or important.

But to many students, especially many philosophy students, logic seems too abstract and mathematical. We study philosophy because we want to think about metaphysics or morality or truth or beauty. Logic prides itself on its lack of content. Moreover, there are rules in logic which can be violated. You can get problems wrong. The solutions to problems are not always obvious.

My advice to students who have difficulty with the computational or mathematical portions of the text is to practice. Do a lot of problems and try to focus on the importance of the tasks at hand. Most importantly, make sure to think about some of the topics in Chapter 4. That chapter is precisely my attempt to make the formal work more interesting and engaging for everyone.

Deduction, Induction, and Ordinary Reasoning

A central question about the logic in this book concerns its relation to ordinary human reasoning. This book focuses on deductive logic. Much of human reasoning is inductive. Deductions are necessary entailments; in a deductive argument, the conclusion follows necessarily from the premises. Inductions are often probabilistic, attempts to gather a variety of evidence into simple, general claims. Perhaps the difference between deduction and induction is best seen by paradigms. DI is a deductive inference; II is an inductive inference.

DI
Polar bears are carnivorous.
Polar bears are mammals.
So, some mammals are carnivorous.

II
47 percent of Americans in a recent poll approve of the way the Supreme Court does its job.
There were 1003 adults polled.
The margin of error for this poll is ±3 percent.
So, between 44 and 50 percent of Americans approve of the way the Court does its work.

Notice that the conclusion of DI is almost disappointing in its obviousness. One you have seen the premises of a deductive argument, the conclusion is rarely surprising. A natural response to the third sentence in DI is, "Yes, you said that already."

In contrast, inductive conclusions can be both surprising and contentious. II is not particularly surprising, but inductive inferences often are, as when a scientist discovers a natural law, or a statistician notices a significant result.

Like ordinary reasoning, much of scientific reasoning is inductive. Indeed, when people say that a conclusion is logical, they often mean that it is well-supported by evidence, not that the conclusion follows necessarily from the premises.

The relation between ordinary reasoning and deductive logic is a topic for debate. Some people, especially some of those who developed modern formal logic over the last 150 years, believe that deductive logic is a normative theory of all serious inference: deductive logic provides the rules that we all must follow to make sure that our inferences are legitimate. Others people believe that deductive logic is mainly uninteresting due to its obviousness. Of course, if you pile enough obvious inferences on top of each other, you get some non-obvious arguments. In this book, you will encounter many such arguments. But, we don't find many complex deductive arguments in our ordinary lives.

The rules of deductive logic are clearly applicable in mathematics and computer science. They may be necessary conditions for all good inferences. But, outside of mathematics, deduction is not sufficient for all our inferences. We make probabilistic judgments constantly. We make observations. We use induction and abduction. We discover facts about the world. Deductive logic may help us organize and frame what we learn, but it does not suffice to account for all of our ordinary reasoning.

So this book has a narrow scope. It is not concerned with the truth of premises. It does not treat the likelihood of inductive inferences. It is a book about deductive consequence. The key ideas of the formal logic at the core of this book were developed in the late nineteenth century. But, logic is a much older discipline. Before starting our formal work, let's look briefly at the history of the discipline and how the contemporary notion of logical consequence was developed.

§2: A Short History of Logic

Aristotle, who lived in the fourth century B.C.E., famously described some fundamental logical rules, called categorical syllogisms. The categorical syllogisms described relations among four kinds of statements, known since the early middle-ages as A, E, I, and O.

A	All Fs are Gs.
E	No Fs are Gs.
I	Some Fs are Gs.
O	Some Fs are not Gs.

In categorical logic, the fundamental elements are terms, portions of assertions. We will look at the modern version of term logic, called predicate or quantificational logic, in Chapter 3.

In the third century B.C.E., the stoic philosopher Chrysippus developed a propositional logic, in which the fundamental elements are complete assertions rather than terms. Some complete assertions are simple, others are complex. Complex assertions are composed of simple assertions combined according to logical rules. In Chapters 1 and 2, we will look at the rules of propositional logic.

Through the middle ages, while there were some major advances in logic, the structure of the discipline was generally stable. After the scientific revolution, philosophers started paying more attention to human psychological capacities. This focus, which we can see in Descartes, Locke, and Hume culminated in the late-eighteenth century work of Kant, and the early nineteenth-century work of Hegel. Kant's logic was essentially a description of how human beings create their experiences by imposing, *a priori*, conceptual categories on an unstructured manifold given in sensation. The term '*a priori*' indicates that Kant believed that some of our intellectual activity is prior to, or independent of, experience. Logic, for Kant, was the description of human psychology, instead of objective rules of consequence. Moreover, according to Kant, logic, as a discipline, was complete.

> We shall be rendering a service to reason should we succeed in discovering the path upon which it can securely travel, even if, as a result of so doing, much that is comprised in our original aims, adopted without reflection, may have to be abandoned as fruitless. That logic has already, from the earliest times, proceeded upon this sure path is evidenced by the fact that since Aristotle it has not required to retrace a single step, unless, indeed, we care to count as improvements the removal of certain needless subtleties or the clearer exposition of its recognised teaching, features which concern the elegance rather than the certainty of the science. It is remarkable also that to the present day this logic has not been able to advance a single step, and is thus to all appearance a closed and completed body of doctrine (Kant, *Critique of Pure Reason* B17).

In the nineteenth century, several developments led mathematicians to worry about logical entailments and to call Kant's claims about logic, its completeness and its psychological status, into question. Since these mathematical worries led directly to the logic in this book, I will take a short detour to discuss two of them: the problem of infinity and non-Euclidean geometries.

For nearly two hundred years, mathematicians had worked with the calculus of Newton and Leibniz. The calculus allowed mathematicians to find the area under a curve by dividing the area into infinitely many infinitely small areas. Working with infinity, both small and large, seemed problematic, even if the resulting calculations were successful. An infinitely small region seemed to be indistinguishable from an empty region. An empty region has zero size. The sum of the sizes of any number of empty regions should still be zero. To make matters worse, Cantor, in the mid-nineteenth century discovered a mathematical proof that there are different sizes of infinity, indeed there are infinitely many difference sizes of infinity. Infinite size had long been identified with God, one of the divine properties in contrast to our human finitude. Cantor's proof struck many mathematicians as absurd, even heretical, but they could not find a flaw in his logic.

Developments in geometry raised similar worries about mathematical inferences. Consider the first four axioms, or postulates, of Euclidean geometry.

E1	Between any two points, one can draw a straight line.
E2	Any straight line segment can be extended indefinitely, to form a straight line.
E3	Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
E4	All right angles are congruent.

Euclid relied on a commonsense interpretation of the terms in these axioms, especially terms for concepts like 'straight' and 'right angle'. Given those ordinary concepts, it seemed obvious that the parallel postulate, Euclid's fifth postulate, would also hold.

E5	If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

E5 is equivalent to Playfair's postulate, PP, which is easier to visualize for some people.

PP	Given a line, and a point not on that line, there exists a single line which passes through the point and is parallel to the given line.

In the two millennia between Euclid and the early nineteenth century, geometers tried in vain to prove E5 or PP. They mainly did so by trying to find that some contradiction would arise from the denials of one or the other. They supposed that there were more than one parallel line through the given point. They supposed that there were no parallel lines through the given point. Both suppositions led to odd kinds of spaces. But, neither supposition led to an outright contradiction.

By the early nineteenth century, some mathematicians realized that instead of leading to contradiction, the denials of E5 and PP lead to more abstract conceptions of geometry, and fecund new fields of study. Riemann and others explored the properties of elliptical geometries, those which arise when adding the claim that there are no parallel lines through the given point in PP to E1-E4. Lobachevsky, Gauss, and others explored the properties of hyperbolic geometries, which arise when adding the claim that there are infinitely many parallel lines through the given point in PP to E1-E4. In both elliptical and hyperbolic geometries, the notions of straightness and right-angularity, among others, have to be adjusted. Our original Euclidean conceptions had been smuggled in to the study of geometry for millennia, preventing mathematicians from discovering important geometric theories.

These geometric theories eventually found important applications in physical science. E5 and PP are equivalent to the claim that the sum of the angles of a triangle is 180°. Consider an interstellar triangle, formed by the light rays of three stars, whose vertices are the centers of those stars. The sum of the angles of our interstellar triangle will be less than 180° due to the curvatures of space-time corresponding to the gravitational pull of the stars and other large objects. Space-time is not Euclidean, but hyperbolic.

As in the case of Cantor's work with infinity, mathematicians considering the counter-intuitive results of non-Euclidean geometries worried that the laws of logical consequence were being flouted. Mathematicians and philosophers began to think more carefully about the notion of logical consequence.

In the late nineteenth century, Gottlob Frege argued that hidden premises, like the assumptions that there is only one size of infinity or that all space must conform to E5, had undermined mathematical progress. Frege wanted to make sure that all branches of mathematics, indeed all of human reasoning, was not liable to similar problems. He thus formalized the study of logical consequence, turning logic into a mathematical subject. In 1879, Frege published *Begriffsschrift*, or *Concept-Writing*, a mathematical logical calculus which subsumed both the term logic of Aristotle and the propositional logic of the stoics. Frege's logic extended and refined the rules of logic, generalizing results.

The preface to Frege's *Begriffsschrift* makes his motivation clear.

> So that nothing intuitive could intrude [into our concept of logical consequence] unnoticed, everything had to depend on the chain of inference being free of gaps. In striving to fulfil this requirement in the strictest way, I found an obstacle in the inadequacy of language: however cumbersome the expressions that arose, the more complicated the relations became, the less the precision was attained that my purpose demanded...The present *Begriffsschrift*...is intended to serve primarily to test in the most reliable way the validity of a chain of inference and to reveal every presupposition that tends to slip in unnoticed, so that its origin can be investigated (Frege, *Begriffsschrift*).

In this book, by separating the syntax of logic, its formation and derivation rules, from its semantics, its interpretations and our ascriptions of truth and falsity, we are attempting to fulfil Frege's dream of a secure theory of logical consequence.

Frege's work, while not immediately recognized as revolutionary, became the foundation for fifty years of intense research in the logical foundations of mathematics and reasoning generally. The culmination of this flurry of research came in the early 1930s with Alfred Tarski's work on truth and Kurt Gödel's incompleteness theorems. Frege's logic, in a neater and more perspicuous form, is mainly the focus of this textbook.

The brief history I just sketched is of course, in its brevity, highly misleading. Many others contributed to the history of logic, especially in the late middle ages. Frege was not the only logician to develop modern logic. Charles Sanders Peirce, for example, independently developed much of what made Frege's logic innovative, his work extending and generalizing Aristotle's categorical logic to include relations. Augustus De Morgan, even earlier than Peirce and Frege, had worked on relational logic. But, Frege's larger project, logicism in the philosophy of mathematics, which I discuss in Chapter 4.10, coming largely as a response to Kant's philosophy and that of the early-nineteenth-century idealists, is especially interesting to contemporary philosophers. Indeed, Frege produced seminal work not only in logic and philosophy of mathematics, but in philosophy of language, epistemology, and metaphysics.

But enough about history. Let's get started with the good stuff.

Chapter 1: Syntax and Semantics for Propositional Logic
§1.1. Separating Premises from Conclusions

The subject of this chapter is the syntax and semantics of propositional logic. Propositional logic is the logic of propositions and their inferential relations. To study propositional logic, we will construct and interpret a language which I will call **PL**. To construct the language, we will specify its syntax. To interpret the language, we will specify its semantics.

Given that our central goal is a better understanding of logical consequence, of what follows from what, our first task must be to look at the ways in which deductive inferences are structured. Compare a disorganized heap of stones with the same pile of stones arranged into the form of a house. The stones are the same. The difference between the two collections is the organizational structure of the latter collection. We want to examine the organizational structure of our inferences.

The basic medium for inference is called an argument. Arguments are sets of propositions used to establish (or derive) one particular claim, the conclusion of the argument. The support for the conclusion, the reasons presented for holding that claim, are called premises. Our first task, then, is to analyze arguments, indicating their structures and separating premises from conclusions. When we analyze an argument, we regiment it to reveal its essential structure.

Here is an argument:

1.1.1 We may conclude that texting while driving is wrong. This may be inferred from the fact that texting is distracting. And driving while distracted is wrong.

The conclusion of this argument is that texting while driving is wrong. The premises are that texting is distracting and that driving while distracted is wrong. In addition to the words used to make those claims, in the original argument, there are premise and conclusion indicators. 'We may conclude that' is used to indicate a conclusion. 'This may be inferred from the fact that' is used to indicate a premise. 'And' is also used to indicate a premise. When we regiment an argument, we eliminate those indicators.

Here are some premise and conclusion indicators:

Premise Indicators	Conclusion Indicators
since	therefore
because	we may conclude that
for	we may infer that
in that	entails that
may be inferred from	hence
given that	thus
seeing that	consequently
for the reason that	so
inasmuch as	it follows that
owing to	implies that
	as a result

While these lists are handy, they should not be taken as categorical. Natural languages like English are inexact and non-formulaic. 'And' often indicates the presence of an additional premise, but can also be used to indicate the extension of a conclusion. Not all sentences in an argument will contain indicators. Often you will have to judge from the content of an argument which propositions are premises and which are conclusions. The best way to determine premises and conclusions is to determine what the main point of an argument is and then look to see what supports that point.

We can regiment 1.1.1. as the perspicuous 1.1.2, eliminating premise and conclusion indicators, and placing the conclusion at the end.

1.1.2 P1. Texting is distracting.
P2. Driving while distracted is wrong.
C. Texting while driving is wrong.

When regimenting an argument, the order of premises is unimportant. 1.1.3 would be just as good a regimentation as 1.1.2.

1.1.3 P1. Driving while distracted is wrong.
P2. Texting is distracting.
C. Texting while driving is wrong.

Similarly, the number of premises is not very important. You can combine or separate premises, though it is often useful to keep the premises as simple as possible. 1.1.4 is logically acceptable, but not as perspicuous as 1.1.2 or 1.1.3.

1.1.4 P1. Driving while distracted is wrong and texting is distracting.
C. Texting while driving is wrong.

The most important task when first analyzing an argument is to determine its conclusion. The most serious mistake you can make in this exercise is to confuse premises and conclusions. Argument 1.1.5 is derived from Leibniz's work.

1.1.5 God is the creator of the world. If this world is not the best of all possible worlds, then either God is not powerful enough to bring about a better world or God did not wish this world to be the best. But God is both omnipotent and all-good. So, this world is the best of all possible worlds.

The central claim of 1.1.5 is that this is the best of all possible worlds. The 'so' at the beginning of the last sentence is a hint to the conclusion. Thinking about the content of the argument should produce the same analysis.

1.1.6 is an unacceptable regimentation of 1.1.5 because it switches a premise and the conclusion. The proper regimentation would switch P3 and C.

1.1.6 P1. God is the creator of the world.
P2. If this world is not the best of all possible worlds, then either God is not powerful enough to bring about a better world or God did not wish this world to be the best.
P3.This world is the best of all possible worlds.
C. God is both omnipotent and all-good.

Sometimes it is not easy to determine how to separate premises from conclusions. Often, such discrimination requires broad context.

Another difficulty arises from single sentences which contain both a premise and a conclusion. Such compound sentences must be divided. 1.1.7 is derived from Locke's work.

1.1.7 Words must refer either to my ideas or to something outside my mind. Since my ideas precede my communication, words must refer to my ideas before they could refer to anything else.

A good regimentation of 1.1.7 divides the last sentence, as in 1.1.8.

1.1.8 P1. Words must refer either to my ideas or to something outside my mind.
P2. My ideas precede my communication.
C. Words must refer to my ideas before they could refer to anything else.

Some arguments contain irrelevant, extraneous information. When constructing an argument, it is better to avoid extraneous claims. Such claims can weaken or even invalidate an argument. When regimenting someone else's argument, it is usually good practice to include all claims, even extraneous ones. Then, when you are evaluating an argument, you can distinguish the important premises from the extraneous ones.

Lastly, some arguments contain implicit claims not stated in the premises. These arguments are called enthymemes. 1.1.9 is enthymemic.

1.1.9 P1. Capital punishment is killing a human being.
C. Capital punishment is wrong.

Again, when regimenting an argument, we ordinarily just show what is explicitly present in the original. When evaluating an argument, we can mention suppressed premises. For instance, we can convert 1.1.9 into a more complete argument by inserting a second premise.

1.1.10 P1. Capital punishment is killing a human being.
P2. Killing a human being is wrong.
C. Capital punishment is wrong.

Notice that P2 here is contentious, which may explain why someone might suppress it. Still, filling out an enthymeme is a job for later, once you have become confident regimenting arguments as they appear.

Exercises 1.1. Regiment each of the following arguments into premise/conclusion form. The inspirations for each argument are noted; not all arguments are direct quotations.

1. Statements are meaningful if they are verifiable. There are mountains on the other side of the moon. No rocket has confirmed this, but we could verify it to be true. Therefore, the original statement is significant. (AJ Ayer, *Language, Truth, and Logic*)

2. It is not only words that are symbolic, but rather, it is things. Everything in nature represents some state of mind. This state of mind can be depicted by presenting its natural appearance as a picture. An enraged man is a lion, a cunning man is a fox, a firm man is a rock, and a learned man is a torch. Distances behind and in front of us are respectively images of memory and hope. (Ralph Waldo Emerson, *Nature*)

3. As humans, we should believe in the theory that best accounts for our sense experience. If we believe in a theory, we must believe in its ontological commitments. The ontological commitments of any theory are the objects over which that theory first-order quantifies. The theory which best accounts for our sense experience first order quantifies over mathematical objects. Then, we should believe that mathematical objects exist. (W.V. Quine)

4. The workingman does not have time for true integrity on a daily basis. He cannot afford to sustain the manliest relations to men, for his work would be minimized in the market. (Henry David Thoreau, *Walden*)

5. It is hard not to verify in our peers the same weakened intelligence due to emotions that we observe in our everyday patients. The arrogance of our consciousness, which in general, belongs to the strongest defense mechanisms, blocks the unconscious complexes. Because of this, it is difficult to convince people of the unconscious and in turn to teach them what their conscious knowledge contradicts. (Sigmund Freud, *The Origin and Development of Psychoanalysis*)

6. The passage from one stage to another may lead to long-continued different physical conditions in different regions. These changes can be attributed to natural selection. Hence, the dominant species are the most diffused in their own country and make up the majority of the individuals, and often the most well marked varieties. (Charles Darwin, *On the Origin of Species*)

7. All of psychology has gotten stuck in moral prejudices and fears. No one has come close to understanding it as the development of the will to power. However, if a person even begins to regard the affects of hatred, envy, covetousness, and the lust to rule as conditions of life and furthermore, as factors essential to the general economy of life, he will begin to get seasick. At this point, he begins to lose himself, and sail over morality. Thus, psychology becomes again the path to fundamental problems. (Friedrich Nietzsche, *Beyond Good and Evil*)

8. Man has no choice about his capacity to feel that something is good or evil. But what he will consider good or evil depends on his standard of value. Man has no automatic knowledge and thus no automatic values. His values are a product of either his thinking or his evasions. (Ayn Rand, *The Virtue of Selfishness*)

9. We must be realists about mathematics. Mathematics succeeds as the language of science. And, there must be a reason for the success of mathematics as the language of science. But, no positions other than realism in mathematics provide a reason. (Hilary Putnam)

10. Local timelines are temporally ordered. The faster you go, the quicker you get to your destination. As you go faster, time itself becomes compressed. But it is not possible to go so fast that you get there before you started. (Albert Einstein, *Relativity*)

11. The sphere is the most perfect shape, needing no joint and being a complete whole. A sphere is best suited to enclose and contain things. The sun, moon, plants, and starts are seen to be of this shape. Thus, the universe is spherical. (Nicolaus Copernicus, *The Revolution of The Celestial Orbs*)

12. The happiest men are those whom the world calls fools. Fools are entirely devoid of the fear of death. They have no accusing consciences to make them fear it. Moreover, they feel no shame, no solicitude, no envy, and no love. And they are free from any imputation of the guilt of sin. (Desiderius Erasmus, *In Praise of Folly*)

13. It is impossible for someone to scatter his fears about the most important matters if he knows nothing about the universe but gives credit to myths. Without the study of nature, there is no enjoyment of pure pleasure. (Epicurus of Samos, *Sovran Maxims*)

14. If understanding is common to all mankind, then reason must also be common. Additionally, the reason which governs conduct by commands and prohibitions is common to us. Therefore, mankind is under one common law and so are fellow citizens. (Marcus Aurelius, *Meditations*)

15. Rulers define 'justice' as simply making a profit from the people. Unjust men come off best in business. But just men refuse to bend the rules. So, just men get less and are despised by their own friends. (Plato, *Republic*)

16. We must take non-vacuous mathematical sentences to be false. This is because we ought to take mathematical sentences at face value. If we take some sentences to be non-vacuously true, then we have to explain our access to mathematical objects. The only good account of access is the indispensability argument. But the indispensability argument fails. (Hartry Field)

17. Labor was the first price, in that it yielded money that was paid for all things. But it is difficult to ascertain the proportion between two quantities of labor. Every commodity is compared with other exchanged commodities rather than labor. Therefore, most people better understand the quantity of a particular commodity, than the quantity of labor. (Adam Smith, *The Wealth of Nations*)

18. Authority comes from only agreed conventions between men. Strength alone is not enough to make a man into a master. Moreover, no man has natural authority over his fellows and force creates no right. (Jean Jacques Rousseau, *The Social Contract*)

19. Mathematics is defined as the indirect measurement of magnitude and the determination of magnitudes by each other. Concrete mathematics aims to discover the equations of phenomena. Abstract mathematics aims to educe results from equations. Therefore, concrete mathematics discovers results by experiment and abstract mathematics derives results from the discovered equations and obtains unknown quantities from known. (Auguste Comte, *The Course in Positive Philosophy*)

20. Just as many plants only bear fruit when they do not grow too tall, so in the practical arts, the theoretical leaves and flowers must not be constructed to sprout too high, but kept near to experience, which is their proper soil. (Carl von Clausewitz, *On War*)

21. The greatest danger to liberty is the omnipotence of the majority. A democratic power is never likely to perish for lack of strength or resources, but it may fall because of the misdirection of this strength and the abuse of resources. Therefore, if liberty is lost, it will be due to an oppression of minorities, which may drive them to an appeal to arms. (Alexis de Tocqueville, *Democracy in America*)

22. There is no distinction between analytic and synthetic claims. If there is an analytic/synthetic distinction, there must be a good explanation of synonymy. The only ways to explain synonymy are by interchangeability *salva veritate* or definition. However, interchangeability cannot explain synonymy. And definition presupposes synonymy. (W.V. Quine)

23. The object of religion is the same as that of philosophy; it is the internal verity itself in its objective existence. Philosophy is not the wisdom of the world, but the knowledge of things which are not of this world. It is not the knowledge of external mass, empirical life and existence, but of the eternal, of the nature of God, and all which flows from his nature. This nature ought to manifest and develop itself. Consequently, philosophy in unfolding religion merely unfolds itself and in unfolding itself it unfolds religion. (Georg Wilhelm Friedrich Hegel, *The Philosophy of Religion*)

24. That the world is my idea is a truth valid for every living creature, though only man can contemplate it. In doing so, he attains philosophical wisdom. No truth is more absolutely certain than that all that exists for knowledge and therefore this world is only object in relation to subject, perception of a perceiver. The world is an idea. (Arthur Schopenhauer, *The World as Will and Idea*)

25. Every art and every inquiry, and similarly every action and pursuit, is thought to aim at some good; and for this reason the good has rightly been declared to be that a which all things aim. (Aristotle, *Nicomachean Ethics*)

26. We should be committed to the entities hypothesized by the mathematics in question. There exist genuine mathematical explanations of empirical phenomena. We should be committed to the theoretical posits hypothesized by these mathematical explanations. (Paolo Mancosu)

27. By 'matter' we are to understand an inert, senseless substance, in which extension, figure, and motion do actually subsist. But it is evident from what we have already shown that extension, figure, and motion are only ideas existing in the mind, and that an idea can be like nothing but another idea, and that consequently neither they nor their archetypes can exist in an unperceiving substance. Hence it is plain that the very notion of what is called matter, or corporeal substance, involves a contradiction in it. (George Berkeley, *A Treatise Concerning the Principles of Human Knowledge*)

28. Reading challenges a person more than any other task of the day. It requires the type of training that athletes undergo, and with the same life-long dedication. Books must be read as deliberately and reservedly as they were written. Thus, to read well, as in, to read books in a true spirit, is a noble exercise. (Henry David Thoreau, *Walden*)

29. Love, friendship, respect, and admiration are the emotional responses of one man to virtues of another, the spiritual payment given in exchange for the personal, selfish, pleasure which one man derives from virtues of another. To love is to value. The man who does not value himself cannot value anyone or anything. (Ayn Rand, *The Virtue of Selfishness*)

30. The only course open to one who wished to deduce all our knowledge from first principles would be to begin with *a priori* truths. An *a priori* truth is a tautology. From a set of tautologies alone, only further tautologies can be further deduced. However, it would be absurd to put forward a system of tautologies as constituting the whole truth about the universe. Therefore, we cannot deduce all our knowledge from first principles. (AJ Ayer, *Language, Truth, and Logic*)

31. Men, in the state of nature, must have reached some point when the obstacles maintaining their state exceed the ability of the individual. Then the human race must either perish or change. Men cannot create new forces, only unite and direct existing ones. Therefore, they can preserve themselves by only combining forces great enough to overcome resistance. (Jean Jacques Rousseau, *On the Social Contract*)

32. Physics can be defined as the study of the laws which regulate the general properties of bodies regarded *en masse*. In observing physics, all senses are used. Mathematical analysis and experiments help with observation. Thus in the phenomena of physics man begins to modify natural phenomena. (Auguste Comte, *The Course in Positive Philosophy*)

33. There are not two indiscernible individuals in our world. If there were two indiscernible individuals in our world then there must be another possible world in which those individuals are switched. God could have had no reason for choosing one of these worlds over the other. But God must have a reason for acting as she does.

34. In aristocratic countries, great families have enormous privileges, which their pride rests on. They consider these privileges as a natural right ingrained in their being, and thus their feeling of superiority is a peaceful one. They have no reason to boast of the prerogatives which everyone grants to them without question. So, when public affairs are directed by an aristocracy, the national pride takes a reserved, haughty and independent form. (Alexis de Tocqueville, *Democracy in America*)

35. It must be some one impression, that gives rise to every real idea. But self or person is not any one impression, but that to which our several impressions and ideas are supposed to have a reference. If any impression gives rise to the idea of self, that impression must continue invariably the same through the whole course of our lives, since self is supposed to exist after that manner. But there is no impression constant and invariable. Pain and pleasure, grief and joy, passions and sensations succeed each other and never all exist at the same time. It cannot, therefore, be from any of these impressions or from any other that the idea of self is derived, and, consequently, there is no idea of the self. (Hume, *A Treatise of Human Nature*)

36. Every violent movement of the will, every emotion, directly agitates the body. This agitation interferes with the body's vital functions. So we can legitimately say that the body is the objectivity of the will. (Arthur Schopenhauer, *The World as Will and Idea*)

37. The work of the defensive forces of the ego prevents repressed desires from entering the conscious during waking life, and even during sleep. The dreamer knows just a little about the meaning of his dreams as the hysteric knows about the significance of his symptoms. The technique of psychoanalysis is the act of discovering through analysis, the relation between manifest and latent dream content. Therefore, the only way to treat these patients is through the technique of psychoanalysis. (Sigmund Freud, *The Origin and Development of Psychoanalysis*)

38. Either mathematical theorems refer to ideal objects or they refer to objects that we sense. If they refer to ideal objects, the radical empiricist cannot defend our knowledge of them, since we never sense such objects. If they refer to objects that we sense, they are false. So for the radical empiricist, mathematical theorems are either unknowable or false. In either case, the radical empiricist cannot justify any proof of a mathematical theorem. (John Stuart Mill)

39. The sense or meaning of a term determines its reference. That is, it is impossible for terms to differ in extension while having the same intension. Reference can vary without variation in thought. So, the senses of terms must be able to vary without variation in thought. So, our thoughts do not determine the meanings of our terms; meanings are not in the head. (Hilary Putnam)

40. My mind is distinct from my body. I have a clear and distinct understanding of my mind, independent of my body. I have a clear and distinct understanding of my body, independent of my mind. Whatever I can clearly and distinctly conceive of as separate, can be separated by God, and so are really distinct. (René Descartes, *Meditations on First Philosophy*)

§1.2: Validity and Soundness

Consider the following three arguments.

1.2.1 P1. All philosophers are thinkers.
P2. Socrates is a philosopher.
C. Socrates is a thinker.

1.2.2 P1. All persons are fish.
P2. Barack Obama is a person.
C. Barack Obama is a fish.

1.2.3 P1. All mathematicians are platonists.
P2. Jerrold Katz is a platonist.
C. Jerrold Katz is a mathematician.

1.2.1 is a good argument for two reasons. First, the conclusion follows from the premises. Second, the premises are true. 1.2.2 and 1.2.3 are both bad arguments, but for different reasons. In 1.2.2, the conclusion follows from the premises, but the first premise is false. In 1.2.3, the premises are true, but the conclusion does not follow from the premises. We call arguments like 1.2.3 invalid. 1.2.2 is valid, but unsound.

The validity of an argument depends on its form. An argument is valid if the conclusion follows logically from the premises. Certain forms of argument are valid. Certain forms are invalid. Most of the first three chapters of this book is dedicated to working with rigorous methods for distinguishing between valid and invalid arguments.

The soundness of a valid argument depends on truth of its premises. A valid argument is sound if all of its premises are true. A valid argument is unsound if any one of its premises are false. Only valid arguments can be sound. (Actually, usage of 'sound' varies. Some people categorize invalid arguments as sound or unsound, but I will not do so.)

Valid arguments are important because in deductive logic, if the form of an argument is valid and the premises are all true, then the conclusion must be true. The previous sentence is the most important sentence of this book. The power of deductive logic is simply that if the premises of an argument in a valid form are true, then, on pain of contradiction, the conclusion of the argument must be true. In invalid arguments, the premises can be true at the same time that the conclusion is false. The central theme of this book, then, is to identify the valid forms of argument.

The validity of an argument is independent of the truth of the premises of an argument. As we saw, 1.2.1 is both valid and sound, while 1.2.2 is valid but unsound. An argument can also have all true premises while being invalid, like 1.2.4.

1.2.4 P1. 2 + 2 = 4.
P2. The sky is blue.
C. Kant wrote the *Critique of Pure Reason*.

Validity is related to possibility, while soundness is related to truth. An argument is valid if it is impossible to make the conclusion false while the premises are true by substituting alternative sentences of the same logical form. This last claim will become a little clearer once we have looked more carefully at the nature of logical form.

Notice that each of the arguments 1.2.5 - 1.2.7 has something in common, which we call its logical form.

1.2.5 Either the stock market will rise or unemployment will go up.
The market will not rise.
So, unemployment will increase.

1.2.6 You will get either rice or beans.
You do not get the rice.
So, you will have the beans.

1.2.7 The square root of two is either rational or irrational.
It is not rational.
So, it's irrational.

We can represent this common logical form by substituting variables for the specific sentences in the argument, using the same variable each time a particular sentence is repeated.

1.2.8 Either P or Q.
Not P.
So, Q.

Just as an architect, when building a building, focuses on the essential structures, so a logician looks mainly at the form of an argument, ignoring the content of the sentences. 'P' and 'Q', above, are variables, standing for statements. 'Either P or Q' is a compound sentence, made of simple ones. We call the form 1.2.8 Disjunctive Syllogism. In Chapter 2, we will identify eight basic valid forms, and use them to determine whether any argument is valid. To start the process of identifying valid forms, we rely on our intuitive judgments about whether some sample inferences are valid or not. The main purpose of the rest of Chapter 1 is to develop the skills for using a rigorous method to determine whether any form is valid.

In our study of propositional logic, we will use capital English letters to stand for simple, positive propositions. Simple propositions are often of subject-predicate form, but not necessarily. They are the shortest examples of statements; they can not be decomposed further in propositional logic. In predicate logic, Chapter 3, we work beneath the surface of propositions.

Exercises 1.2. Determine whether each of the following arguments is intuitively valid or invalid. For valid arguments, determine, as best you can, whether they are sound.

1. Obama and McCain were the only two major-party candidates in the 2008 election. A major party candidate won the election. McCain did not win. So, Obama won the 2008 election.

2. Either Obama wins the 2012 presidential election or Romney does. If Obama wins, all of California will be happy. If Romney wins, all of Utah will be happy. So either all of California is happy or all of Utah is happy.

3. Only men have been presidents of the United States. All presidents have been wealthy. Obama is a man. Therefore, Obama is wealthy.

4. Only citizens can run for president. All citizens who have run for president have been male. Bill Clinton is a male citizen. Hence, he ran for president.

5. Only citizens can be president of the United States. Only men have been president. So, only male citizens have been president.

6. Frankfort is the capital of Kentucky. Trenton is the capital of New Jersey. Phoenix is the capital of Arizona. It follows that Raleigh is the capital of North Carolina.

7. All princesses are women. Kate Middleton is a princess. Therefore, Kate Middleton is a woman.

8. All horses are mammals. All horses have four legs. So, all mammals have four legs.

9. All unicorns are pink. All unicorns have horns. If something is pink and has a horn, it must be a unicorn.

10. Archaeologists are anthropologists. Anthropologists are social scientists. It follows that archaeologists are social scientists.

11. All trees are tall. All tall things are hard to climb. So, all trees are hard to climb.

12. Either all cats are black or all cats are fluffy, but no cat is both. My cat is fluffy. So, my cat is not black.

13. Either some cats are black or all cats are fluffy. All cats are black. So, some cats are fluffy.

14. Some cats are black. Some cats are fluffy. So, some cats are black and fluffy.

15. Some cats are fluffy. All cats have whiskers. So all fluffy cats have whiskers.

16. Physics and psychology are sciences. Psychologists are smarter than physicists. Therefore, psychology is better than physics.

17. All windows are made of glass. Glass is transparent. So all windows must be transparent.

18. All penguins are birds. All birds have wings. All winged creatures can fly. So it follows that penguins can fly.

19. All eagles are birds. Eagles are endangered species. So, birds are endangered species.

20. If it is sunny, then the lacrosse team practices outside. It is sunny. So, the lacrosse team practices outside.

21. Either it is raining or it is sunny, but not both. It is not raining. So, it is sunny.

22. Either I stop smoking or I risk getting ill. If I stop smoking, then I will have withdrawal symptoms. If I get ill, then I risk death. So either I have withdrawal symptoms or I risk death.

23. All humans breathe. Bob breathes. Hence, Bob is human.

24. All guitars are instruments. All guitars have strings. So, all instruments have strings.

25. All fish live in the Atlantic Ocean. The Atlantic Ocean is a body of water. So, all fish live in a body of water.

26. Any papers that receives an A has a thesis. Wanda's paper receives an A. Hence, her paper has a thesis.

27. All rats have tails. Some rats are white. So, all rats are white and have tails.

28. All rats have tails. Some rats are white. Therefore, all white rats have tails.

29. All rats are white. Only white rats are used in genetic experiments. A rat is used in a genetic experiment. So, that rat must be white.

30. All squares are rectangles. All rectangles are parallelograms. All parallelograms are quadrilaterals. Therefore, all squares are quadrilaterals.

§1.3: Logical Connectives and Translation

Natural languages, like English, as well as many formal languages, have a finite stock of simple, or atomic, sentences and an infinite number of grammatically correct sentences. To produce complex sentences from simple ones, we use what the grammarian calls conjunctions and what logicians call connectives. The principle by which we can compose longer sentences from shorter ones is called compositionality.

In classical logic, we assume compositionality: we can construct sentences of any length. In natural language, we usually find it convenient to divide our discourse into small sentences. But, compositionality is evident, to some degree, in natural language too. Consider this passage from a much-longer story composed of a single sentence.

> Now they're going to see who I am, he said to himself in his strong new man's voice, many years after had seen the huge ocean liner without lights **and** without any sound which passed by the village one night like a great uninhabitated palace**,** longer than the whole village and much taller than the steeple of the church, **and** it sailed by in the darkness toward the colonial city on the the other side of the bay that had been fortified against buccaneers**,** with its old slave port and the rotating light, whose gloomy beams transfigured the village into a lunar encampment of glowing houses **and** streets of volcanic deserts every fifteen seconds… (Gabriel García Márquez, "The Last Voyage of the Ghost Ship", empahses added).

Grammarians usually bristly at long, run-on sentences. But, from a logical point of view, we can build sentences of indefinite length by repeated applications of connectives. In logic, we reserve the term 'conjunction' for a particular one of the various logical connectives. The system of propositional logic that we will study uses five connectives, which we identify by their syntactic properties, or shapes.

Tilde	~
Dot	•
Wedge	∨
Hook	⊃
Triple-bar	≡

These connectives are used to represent certain logical operations of conjoining simple sentences. We will consider five basic logical operations, though systems of logic can be built from merely one or two operations. We could also introduce other, less-intuitive logical operations. These five are standard.

Negation	~
Conjunction	•
Disjunction	∨
Material Implication	⊃
Material Biconditional	≡

What follows is a more detailed explication of each of our five connectives.

Negation

Negation is a unary operator, which means that it applies to a single proposition. The other four operators are all binary, relating two propositions. Some English indicators of negation include:

Not
It is not the case that
It is not true that
It is false that

Each of the three sentences following the proposition 1.3.1 expresses a negation of it.

1.3.1 John will take the train
1.3.2 John won't take the train.
1.3.3 It's not the case that John will take the train.
1.3.4 John takes the train...not!

We can represent 1.3.1 as 'P' and each of 1.3.2 - 1.3.4 as '~P'.

A proposition is called a negation if its main operator is a negation. 1.3.5 - 1.3.7 are all negations.

1.3.5 $\sim R$
1.3.6 $\sim(P \bullet Q)$
1.3.7 $\sim\{[(A \vee B) \supset C] \bullet \sim D\}$

Conjunction

Some English indicators of a logical conjunction are:

and
but
also
however
yet
still
moreover
although
nevertheless
both

Here are some English sentences which we can represent as conjunctions.

1.3.5	Angelina walks the dog and Brad cleans the floors.	$A \bullet B$
1.3.6	Although Angelina walks the dog, Brad cleans the floors.	$A \bullet B$
1.3.7	Bob and Ray are comedians.	$B \bullet R$
1.3.8	Carolyn is nice, but Emily is really nice.	$C \bullet E$

While the logical operator in each of 1.3.5 - 1.3.8 is a conjunction, the tone of the conjunction varies. Logicians often distinguish between the logical and pragmatic properties of language. 'And' and 'but' are both used to express conjunctions even though they have different practical uses.

We use conjunctions to combine complete sentences. In English, 1.3.7 is short for a more complete sentence like 1.3.9.

1.3.9 Bob is a comedian and Ray is a comedian.

Sometimes, sentences using 'and' are not naturally rendered as conjunctions.

1.3.10 Bob and Ray are brothers.

1.3.10 is most naturally interpreted as expressing a relation between two people, and not a conjunction of two sentences. Of course, 1.3.10 could also be used to express the claim that both Bob and Ray are monks, in which case it would be best logically represented as a conjunction. In propositional logic, we regiment the most natural sense of 1.3.10 merely as a simple letter: 'P', say. We will explore the latter sense in Chapter 3. The difference between the two interpretations can not be found in the sentence itself. It has to be seen from the use of the sentence in context. Many sentences are ambiguous when seen out of context. In symbols, 1.3.11 - 1.3.13 are all conjunctions.

1.3.11 $P \bullet {\sim}Q$
1.3.12 $(A \supset B) \bullet (B \supset A)$
1.3.13 $(P \vee {\sim}Q) \bullet {\sim}[P \equiv (Q \bullet R)]$

Disjunction

Some English indicators of disjunction include:

or
either
unless

Most disjunctions use an 'or'. 'Unless' is tricky; see Chapter 4, §2, for a discussion of its subtleties. 1.3.14 - 1.3.16 are English sentences which we can represent as disjunctions.

1.3.14	Either Paco makes the Website, or Matt does.	$P \vee M$
1.3.15	Jared or Rene will go to the party.	$J \vee R$
1.3.16	Justin doesn't feed the kids unless Carolyn asks him to.	$J \vee C$

In symbols, all of the following are disjunctions:

1.3.17 ${\sim}P \vee Q$
1.3.18 $(A \supset B) \vee (B \supset A)$
1.3.19 $(P \vee {\sim}Q) \vee {\sim}[P \equiv (Q \bullet R)]$

Material Implication (The Conditional)

Some English indicators of material implication are:

if
only if
only when
is a necessary condition for
is a sufficient condition for
implies
entails
provided that
given that
on the condition that
in case

Material implication is sometimes called the material conditional. As we will see, it is not a perfect representation of the natural-language conditional. But there seem to be no better ways to represent the conditional logically. Some functions of the natural language conditional are not logical. Material implication is the best logical form for the natural language conditional, and using material implication allows us to regiment some conditional sentences effectively.

When using material implication, unlike with disjunctions and conjunctions, the order of the related propositions is important. In ‘A ⊃ B’, ‘A’ is called the antecedent and ‘B’ is called the consequent. While we’re defining terms, given a conditional ‘A ⊃ B’, we can define three related conditionals. ‘B ⊃ A’ is called the converse; ‘~A ⊃ ~B’ is called the inverse; and ‘~B ⊃ ~A’ is called the contrapositive. A statement and its contrapositive are logically equivalent. The inverse and the converse of a conditional are logically equivalent to each other. But, a conditional is not equivalent to either its inverse or its converse. I will explain those results and what ‘logical equivalence’ means in §1.6.

Here are some examples of natural language conditionals and their translations into propositional logic, using ‘A’ to stand for ‘you join me’ and ‘B’ to stand for ‘I go to the movies’.

1.3.20	If you join me, then I go to the movies.	If A then B	A ⊃ B
1.3.21	You join me if I go to the movies.	If B then A	B ⊃ A
1.3.22	You join me only if (only when) I go to the movies.	A only if (only when) B	A ⊃ B
1.3.23	Your joining me is a necessary condition for my going.	A is necessary for B	B ⊃ A
1.3.24	Your joining me is a sufficient condition for my going.	A is sufficient for B	A ⊃ B
1.3.25	A necessary condition of your joining me is my going.	B is necessary for A	A ⊃ B
1.3.26	A sufficient condition for your joining me is my going.	B is sufficient for A	B ⊃ A
1.3.27	Your joining me entails (implies) that I go to the movies.	A entails (implies) B	A ⊃ B
1.3.28	You join me given (provided, on the condition) that I go.	A given (provided, on the condition) B	B ⊃ A

Note that necessary conditions are consequents, while sufficient conditions are antecedents. The case of sufficient conditions is fairly easy to understand. Necessary conditions are trickier. If A is necessary for B, then if B is true, we can infer that A must also be true. For example, some sort of exercise is necessary for good physical health. We can thus claim that if someone is healthy, then he or she must do some sort of exercise; the necessary condition is in the consequent. But, we can not thus claim that if someone exercises, he or she must be physically healthy. There might be other conditions impeding such a person's health. To remember that sufficient conditions are antecedents and necessary conditions are consequents, we can use the mnemonic 'SUN'. Rotating the 'U' to a '⊃' we get 'S ⊃ N'.

In symbols, all of the following are conditionals.

1.3.29	$\sim P \supset Q$
1.3.30	$(A \supset B) \supset (B \supset A)$
1.3.31	$(P \vee \sim Q) \supset \sim[P \equiv (Q \bullet R)]$

The Material Biconditional

Biconditional statements are really conjunctions of two conditional statements. Some English indicators of a biconditional include:

if and only if
is a necessary and sufficient condition for
just in case.

The biconditional '$A \equiv B$' is short for '$(A \supset B) \bullet (B \supset A)$', to which we will return, once we are familiar with truth conditions. Here is an English example of a biconditional.

1.3.32	You'll be successful just in case you work hard and are lucky.	$S \equiv (W \bullet L)$

In symbols, all of the following are biconditionals:

1.3.33	$\sim P \equiv Q$
1.3.34	$(A \supset B) \equiv (B \supset A)$
1.3.35	$(P \vee \sim Q) \equiv \sim[P \equiv (Q \bullet R)]$

Ambiguous Cases

Now that you have seen each of the five connectives and their English language approximations, you can start to translate both simple and complex English sentences into propositional logic. Given a translation key, you can also interpret sentences of propositional logic as English sentences. When translating between English and propositional logic, make sure to resolve or avoid ambiguities.

1.3.33	You may have salad or potatoes and carrots.
1.3.34	$(S \vee P) \bullet C$
1.3.35	$S \vee (P \bullet C)$

We might translate 1.3.33 as 1.3.34, but, we might translate it as 1.3.35. There is an important difference between the two translations. In the first case, you are having carrots and either salad or potatoes. In the second case, you are either having one thing (salad) or two things (potatoes and carrots). To avoid ambiguities, look for commas and semicolons.

1.3.36 You may have salad or potatoes, and carrots.
1.3.37 You may have salad, or potatoes and carrots.

With commas, 1.3.36 is clearly best translated as 1.3.34, while 1.3.37 is clearly best translated as 1.3.35.

Exercises 1.3a. Translate each sentence into propositional logic using any obvious letters.

1. Andre likes basketball.
2. Andre doesn't like soccer.
3. Pilar and Zach are logicians.
4. Sabrina wants either a puppy or a kitten.
5. Kangaroos are marsupials and they live in Australia.
6. Pablo will go to the store if and only if his brother drives him and pays for gas.
7. Everybody loves Raymond, or not.
8. Brittany likes fish and lizards, but not cats.
9. If Beth rides her bike, she gets to work earlier.
10. José cooks only when his mother comes over for dinner.
11. Martina doesn't like shopping unless Jenna comes with her.
12. The world will end just in case aliens invade.
13. It is safe to swim if and only if the water is calm or a lifeguard is on duty.
14. Doctors are helpful unless their patients are rude.
15. Logic is challenging and fun given that you pay attention in class.
16. Turtles live long lives and are happy creatures, unless they are harmed.
17. Elliott wants to skateboard or write songs and play them.
18. Cars are eco-friendly if they are hybrids or run on low-emission fuel.
19. Dylan doesn't like math or science.
20. Cara will go horseback riding only if it doesn't rain and she has a helmet.

Exercises 1.3b. Interpret the following sentences of propositional logic using the given translation key. Strive for elegance in your English sentences.

A: Louisa teaches English.
B: Louisa teaches history.
C: Louisa teaches in a middle school.
D: Louisa has a Master's degree.
E: Javier teaches English.
F: Suneel teaches English.

1. C ⊃ (B ∨ A)
2. A • ~B
3. A ⊃ (E • F)
4. ~D ⊃ ~(A ∨ B)
5. ~(E ∨ F) ⊃ B

G: Jeremy majors in philosophy.
H: Jeremy majors in physics.
I: Jeremy majors in psychology.
J: Jeremy is a college student.
K: Marjorie is a philosophy professor.
L: Marjorie teaches logic.

6. (K • L) ⊃ G
7. J ⊃ (G • I)
8. ~(G • I) ∨ ~H
9. ~(K • L) ⊃ (I ∨ H)
10. G ≡ (J • K)

M: Carolina plants vegetables.
N: Carolina plants flowers.
O: Carolina has a garden.
P: Carolina's plants grow.
Q: Carolina sprays her plants with pesticides.
R: Deer eat the plants.

11. O ⊃ (M • N)
12. (O • P) ⊃ R
13. [(N • P) • Q] ⊃ ~R
14. [(M ∨ N) • P] ⊃ (Q ∨ R)
15. ~P ≡ ~ Q

§1.4. Syntax of **PL**: Wffs and Main Operators

To this point, we have been rather causal about the language of propositional logic. I will now be more rigorous in setting up the syntax of our first formal language of the course, one of many different languages for propositional logic. I will call our language **PL**. To specify a formal language, we start with a list of the vocabulary of the language, its formal symbols. For our purposes, the following thirty-seven different symbols will suffice.

Capital English letters, used as propositional variables	A ... Z
Five connectives:	~, •, ∨, ⊃, ≡
Punctuation:	(,), [,], {, }

Notice that **PL** contains only 26 propositional variables. More flexible systems of propositional logic can accommodate infinitely many propositional variables by including the prime symbols among its vocabulary and allowing iterated repetitions of it.

P, P', P'', P''', P'''', P'''''...

Since we won't be needing so many variables, we will just use English letters with no primes. But keep in mind that we could use just one letter and one prime symbol to get infinitely many variables.

Once we have specified the vocabulary of a formal language, we can start to put these symbols together. Consider the following two strings of English letters.

1.4.1 baker
1.4.2 aebkr

1.4.1 is a well-formed English expression, a word, and 1.4.2 is not. Analogously, in our language of propositional logic, only some strings of symbols are well-formed. We call strings of logical symbols which are constructed properly well-formed formulas, or wffs. 'Wff' is pronounced like 'woof', as if you are barking. 1.4.3 and 1.4.4 are wffs, while 1.4.5 and 1.4.6 are not wffs.

1.4.3 P • Q
1.4.4 (~P ∨ Q) ⊃ ~R
1.4.5 • P Q
1.4.6 Pq ∨ R~

In English, we can determine if a string of letters is a word by looking at a list of English words, as in a dictionary. Such a list is very long, containing perhaps over a million entries. But, it is a finite, if growing, list. In propositional logic, in contrast, we have infinitely many wffs. Wffs of indefinite length may be constructed by applying a simple set of rules, called formation rules.

Formation rules for wffs of PL

1. A single capital English letter is a wff.
2. If α is a wff, so is $\sim\alpha$.
3. If α and β are wffs, then so are:
   $(\alpha \bullet \beta)$
   $(\alpha \vee \beta)$
   $(\alpha \supset \beta)$
   $(\alpha \equiv \beta)$
4. These are the only ways to make wffs.

By convention, we may drop the outermost brackets that automatically appear when forming a wff using Rule 3. Those brackets must be replaced when a shorter formula is included in a more-complex formula. 1.4.7 provides an example of how one might construct a complex wff using the formation rules.

1.4.7	W	By rule 1
	X	By rule 1
	~W	By rule 2
	~W • X	By rule 3, and the convention for dropping brackets
	(~W • X) ≡ ~X	By rules 2 and 3, putting the brackets back
	~[(~W • X) ≡ ~X]	By rule 2

The last connective added according to the formation rules is called the main operator. We can determine the main operator of any wff of **PL** by analyzing the formation of that wff, as I do at 1.4.8.

1.4.8 (~M ⊃ P) • (~N ⊃ Q)

‘M’, ‘P’, ‘N’, and ‘Q’ are all wffs, by rule 1.
‘~M’ and ‘~N’ are wffs by rule 2.
‘(~M ⊃ P)’ and ‘(~N ⊃ Q)’ are then wffs by rule 3.
Finally, the whole formula is a wff by rule 3 and the convention for dropping brackets.

Exercises 1.4a. Are the following formulas wffs? If so, which connective is the main operator?

1. $C \supset D \bullet E$
2. $(T \bullet V) {\sim} W$
3. $(J \vee {\sim} J) \supset K$
4. ${\sim}[(A \vee B) \supset C]$
5. ${\sim}(A \bullet B) \supset C \vee D$
6. $(P \bullet Q \vee R) \supset {\sim} S$
7. ${\sim}\ (K \supset N) \supset (L \bullet M)$
8. ${\sim} K\ [M \equiv (N \bullet O)]$
9. $(D \vee E) \equiv {\sim}[(F \supset G) \bullet H)]$
10. $[D \supset (E \bullet F)] \vee (F \equiv D)$
11. $(S \vee L) \supset C \supset (Q \bullet R)$
12. $(X \vee Y) \equiv [(Y \vee Z) \bullet (X \vee Z)]$
13. $(X \bullet Y {\sim} Z) \supset [(F \vee {\sim} G) \equiv {\sim} H]$
14. ${\sim}\{(P \bullet Q) \supset [(P \bullet R) \vee (R \supset Q)]\}$
15. $[(T \vee U) \bullet (U \vee V)] \supset [(V \bullet W) \vee (T \bullet W)]$

Exercises 1.4b. Translate these sentences into propositional logic using any obvious letters.

1. The restaurant served chicken, and either peas or carrots.

2. Making butter is a necessary condition for the farmer to go to the market and make a profit.

3. Patrons may have corn and potatoes if and only if they do not order carrots.

4. The restaurant serves pie or cheesecake or ice cream for dessert.

5. If the restaurant runs out of cheesecake, then you can have a meal of chicken and pie and ice cream.

6. A farmer keeps goats in a pen and sheep in a pen only if the dogs and cat are kept inside.

7. Either the farmer shears the sheep and milks the cows, or he slops the pigs and walks the dogs.

8. If the farmer shears the sheep, then he makes wool, and if he milks the cows, then he makes butter.

9. The restaurant doesn't have peas, so instead they serve corn and potatoes.

10. If the farmer goes to the market, then he makes a profit, and his wife is happy.

11. Plato believed in the theory of forms and Aristotle held that there are four kinds of causes, but Parmenides thought that only the one exists.

12. If Thales reduced everything to water, then Democritus was an atomist if and only if Heraclitus claimed that the world is constantly in flux.

13. If Plato believed in the theory of forms or Democritus was an atomist, then Aristotle held that there are four kinds of causes or Parmenides thought that only the one exists.

14. Democritus was not an atomist if and only if Plato didn't believe in the theory of forms and Thales didn't reduce everything to water.

15. Either Heraclitus claimed that the world is constantly in flux or Thales reduced everything to water, and either Aristotle held that there are four kinds of causes or Parmenides thought that only the one exists.

16. Smart believes that minds are brains and Skinner thinks that inner states are otiose, unless Descartes argues that the mind and body are distinct.

17. Either Putnam claims that minds are probabilistic automata, or the Churchlands deny that there are any minds and Turing believes that machines can think.

18. Searle rejects the possibility of artificial intelligence if and only if Smart believes that minds are brains and Turing believes that machines can think.

19. Putnam doesn't claim that minds are probabilistic automata and the Churchlands don't deny that there are any minds, if Skinner thinks that inner states are otiose, or Searle rejects the possibility of artificial intelligence and Descartes doesn't argue that the mind and body are distinct.

20. Either Turing believes that machines can think or Smart doesn't believe that minds are brains, and the Churchlands deny that there are any minds.

§1.5. Truth Functions

When constructing a formal system of logic, we start with a language like **PL**. In §1.4 I provided formation rules, or a syntax, for that language. Once we have specified the language, there are two ways that we can use it. First, we can interpret the language, providing a semantics for it. Second, we can use the language in a deductive system by introducing transformation and/or inference rules. We will study inference rules in Chapter 2. Here, we will discuss the interpretations, or semantics, of our language.

Informally, we might interpret some of our propositional variables as particular English propositions. For example, we might take 'P • Q' to stand for 'It is both raining and snowing in Clinton NY right now'. More formally, and more generally, in **PL** and all standard propositional logics, we interpret propositional variables by assigning truth values to them. We will study a bivalent logic, which means that we will consider only two truth values: truth and falsity. Other systems of logic use three or more truth values.

We have carefully circumscribed our language **PL**. It does not contain tools for doing the interpretation. To interpret our formal language, we use a metalanguage. Our metalanguage will be English, supplemented with some specific symbols used with specific intents. For example, we will use '1' to represent truth and '0' to represent falsity. We specify rules of our metalanguage much less formally.

We will start our study of the semantics of propositional logic by looking at how we calculate the truth value of a complex proposition on the basis of the truth values of its component sentences. We can calculate the truth value of any complex proposition using the truth values of its component propositions and the basic truth tables for each connective. The fact that the truth values of complex propositions are completely determined by the truth values of the component propositions is called truth-functional compositionality. Compositionality is a basic presupposition of our logic. Consider a complex proposition like 1.5.1 and its translation into **PL** 1.5.2.

1.5.1 Either *The Hurt Locker* or *Avatar* won the 2010 Oscar for Best Picture, but *The Hurt Locker* won if, and only if, George Clooney did not win the 2010 Oscar for Best Actor.

1.5.2 (H ∨ A) • (H ≡ ~C)

We know the truth values of the component propositions H, A, and C: H is true because *The Hurt Locker* won the Oscar. A is false, since *Avatar* did not win the Oscar. C is false because George Clooney did not win the Oscar. But what is the truth value of the whole complex proposition 1.5.1?

The *truth value of a complex proposition* is the truth value of its main operator.

To determine the truth value of a complex proposition, we combine the truth values of the component propositions using rules for each connective. These rules are summarized in basic truth tables, one for each connective.

Once we combine these truth tables, our semantics, with our translations of natural languages into **PL**, certain problems arise. Not all of our natural-language sentences conform precisely to the semantics given by the truth tables. Difficulties arise for the conditional, in particular. In this section, I discuss the details of the truth tables for each connective and how to use the basic truth tables. In Chapter 4, especially §2 and §3, I discuss some of the more interesting philosophical questions which arise from our use of the standard semantics I present here.

Negation

1.5.3 Two plus two is four.
1.5.4 Two plus two is not four.
1.5.5 Two plus two is five.
1.5.6 Two plus two is not five.

Note that while 1.5.3 is true, its negation, 1.5.4, is false. Also, while 1.1.5 is false, its negation, 1.5.6, is true. We generalize these results using a truth table. In the first row of the truth table, we have a connective, the tilde, and a Greek letter, α. We will use Greek letters as metalinguistic variables which stand for any object-language propositional variable. The column under the 'α' represents all possible assignments of truth values to a single proposition. The column under the '~' represents the values of the negation of that proposition in each row. A truth table for a complex proposition containing one variable has two lines, since there are only two possible assignments of truth values. This truth table says that if the value of a propositional variable is true, the value of its negation is false, and if the value of a propositional variable is false, the value of its negation is true.

~	α
0	1
1	0

Conjunction

1.5.7 He likes logic and metaphysics.

1.5.7 is true if 'He likes logic' is true and 'He likes metaphysics' is true. It is false otherwise. Note that we need four lines to explore all the possibilities of combinations of truth values of two propositions: when both are true, when one is true and the other is false (and vice-versa), and when both are false.

α	•	β
1	1	1
1	0	0
0	0	1
0	0	0

Our basic truth tables all have either two lines or four lines, since all of our connectives use either one or two variables. Truth tables for more-complex sentences, can be indefinitely long. Truth tables with three variables require eight lines. With four variables we need sixteen lines. More generally, truth tables for *n* variables require 2^n lines. We will construct larger truth tables in §1.6.

Disjunction

1.5.8 She can get an A in either history or physics.

We are going to use an inclusive disjunction, on which 1.5.8 is false only when both component statements are false.

α	$\vee$	β
1	1	1
1	1	0
0	1	1
0	0	0

There is an alternate use of 'or' on which a disjunction is false also when both of its component propositions is false.

1.5.9 You may have either soup or salad.

Uses of 1.5.9 are usually made to express that one may have either soup or salad, but not both. This latter use of 'or' is called exclusive disjunction. We can define exclusive disjunction in terms of inclusive disjunction with the help of other connectives. So, it does not matter whether we take inclusive or exclusive disjunction as the semantics of $\vee$ as long as we are clear about what we mean when we are regimenting natural language sentences into our formal logic.

We will use inclusive disjunction, the $\vee$, to translate 'or' unless there is a good reason to switch to exclusive disjunction. See Chapter 4, §2, for a more-detailed discussion of exclusive disjunction.

Material Implication

1.5.10 If you paint my house, then I will give you $5000.

To interpret English-language conditionals, we use what is called the material interpretation, represented by the truth table for $\supset$.

α	$\supset$	β
1	1	1
1	0	0
0	1	1
0	1	0

To understand the material interpretation, consider when 1.5.10 will be falsified. It is true in the first row, when both the antecedent and consequent are true. It is false in the second row, when the antecedent is true and the consequent is false. In the third and fourth rows, when the antecedent is false, we consider 1.5.10 as unfalsified, and thus true. We thus treat a conditional with a false antecedent as an open, and therefore true, sentence. Since you haven't painted my house, 1.5.10 is true whether or not I give you $5000. The only case in which 1.5.10 is clearly false is when you paint my house and I fail to give you the money; that's the second row of the truth table for ⊃.

The conditional is the trickiest connective, in large part because many of our uses of 'if...then...' are not truth-functional. In other words, the truth value of many complex sentences which use conditionals are not exclusively dependent on the truth values of their components.

1.5.11 If this sugar cube is dropped into a pot of warm water, then it will dissolve.
1.5.12 If this piece of steel is dropped into a pot of warm water, then it will dissolve.

We naturally believe that 1.5.11 is true and 1.5.12 is false. The sentences depend for their truth not merely on the truth values of the component propositions, but on the laws of physics. But, they have the same truth conditions as far as ⊃ is concerned. We treat both sentences as true if the sugar cube and the steel are never dropped into water.

Still, some uses of conditionals in English are truth-functional and we are going to use '⊃' to regiment conditionals into **PL**. For a more-detailed discussion of the deep problem of how to understand natural-language conditionals, see Chapter 4, §3.

The Material Biconditional

A biconditional is true if the component statements share the same truth value. It is false if the components have different values.

1.5.13 Supplies rise if and only if demand falls.

α	≡	β
1	1	1
1	0	0
0	0	1
0	1	0

If supplies rise and demand falls, 1.5.13 is true. If supplies don't rise and demand doesn't fall, then 1.5.13 is true as well. But if one happens without the other, then 1.5.13 is false.

The material biconditional is thus really a shorthand for two material conditionals: if α then β and if β then α. The result is that ≡ works like an equals sign for propositions: it will be true if and only if the truth values of the components are the same.

Truth Values of Complex Propositions

The basic truth tables can be used to evaluate the truth value of any proposition built using the formation rules.

Method for Determining the Truth Value of a Proposition

1. Assign truth values to each simple formula.
2. Evaluate any negations of those formulas.
3. Evaluate any connectives for which both values are known.
4. Repeat steps 2 and 3, working inside out, until you reach the main operator.

Remember, the truth value of a complex proposition is the truth value of its main operator. Using this method, we can evaluate the truth value for any sentence as long as we know the truth values of its component propositions. Consider 1.5.14.

1.5.14 (A ∨ X) • ~B

Let's arbitrarily assume that A and B are true and X is false. If we were starting with an English sentence, we might be able to determine appropriate truth values of the component sentences.

First, assign the values to A, B, and X:

(A	∨	X)	•	~	B
1		0			1

Next, evaluate the negation of B:

(A	∨	X)	•	~	B
1		0		1	1

Since you know the values of the disjuncts, you can next evaluate the disjunction:

(A	∨	X)	•	~	B
1	1	0		0	1

Finally, you can evaluate the main operator, the conjunction:

(A	∨	X)	•	~	B
1	1	0	**0**	0	1

1.5.14 is thus false for the values we arbitrarily assumed.

Let's return to 1.5.2. We can assign values to 'H', 'A', and 'C' since we already knew them. Then, we can use our method for determining the truth value of a complex proposition.

(H	∨	A)	•	(H	≡	~	C)
1	1	0	**1**	1	1	1	0

1.5.2 is thus true. 1.5.15 and 1.5.16 are further examples.

1.5.15 A ⊃ (~X • ~Y) where A is true and X and Y are false

A	⊃	(~	X	•	~	Y)
1	**1**	1	0	1	1	0

1.5.15 is true for our assumed values.

1.5.16 [(A • B) ⊃ Y] ⊃ [A ⊃ (C ⊃ Z)] where A, B, and C are true; Y and Z are false.

[(A	•	B)	⊃	Y]	⊃	[A	⊃	(C	⊃	Z)]
1	1	1	0	0	**1**	1	0	1	0	0

1.5.16 is true for the given assignments of truth values.

Complex Propositions with Unknown Truth Values

We have seen how to calculate the truth value of a complex proposition when the truth values of the components are known. Sometimes you don't know truth values of one or more component variable. If the truth values come out the same, whatever values we assign, then the statement has that truth value. If the values come out different in different cases, then the truth value of the complex statement is really unknown.

Consider 1.5.17 and suppose that A, B, C are true; X, Y, Z are false; and P and Q are unknown.

1.5.17 P • A

If P were true, then the truth value of 1.5.17 would be true.

P	•	A
1	**1**	1

If P were false, then 1.5.17 would be false.

P	•	A
0	**0**	1

Since the truth value of 1.5.17 depends on the truth value of P, it too is unknown. Sometimes, in contrast, we can determine the truth value of a complex proposition even when the truth values of one or more of the component propositions is unknown.

1.5.18 P ∨ A

If P is true, then 1.5.18 is true.

P	∨	A
1	**1**	1

If P is false, then 1.5.18 is true too!

P	∨	A
0	**1**	1

The truth value of 1.5.18 is true in both cases. In our bivalent logic, these are the only cases we have to consider. Thus, the value of that statement is true, even though we didn't know the truth value of one of its component propositions. We have seen that the truth value of a complex proposition containing a component proposition with an unknown truth value may be unknown and it may be true. Sometimes the truth value of such a complex proposition will come out false, like 1.5.19.

1.5.19 Q • Y

If Q is true, then 1.5.19 is false

Q	•	Y
1	**0**	0

If Q is false, then 1.5.19 is also false.

Q	•	Y
0	**0**	0

Since the truth value of the complex proposition is false in both cases, the value of 1.5.19 is false.

Lastly, we can have more than one unknown in a statement. If there are two unknowns, we must consider four cases: when both propositions are true, when one is true and the other is false, the reverse case, and when both are false, as in 1.5.20.

1.5.20 $(A \supset P) \lor (Q \supset A)$ where A is true

(A	⊃	P)	∨	(Q	⊃	A)
1	1	1	**1**	1	1	1
1	1	1	**1**	0	1	1
1	0	0	**1**	1	1	1
1	0	0	**1**	0	1	1

Since all possible substitutions of truth values for ‘P’ and ‘Q’ in 1.5.20 yield a true statement, the statement itself is true.

Exercises 1.5a. Assume A, B, C are true and X, Y, Z are false. Evaluate the truth values of each:

1. $X \lor Z$
2. $A \bullet {\sim}C$
3. ${\sim}C \supset Z$
4. $(A \bullet Y) \lor B$
5. $(Z \equiv {\sim}B) \supset X$
6. $(A \supset B) \lor {\sim}X$
7. $(Z \bullet {\sim}X) \supset (B \lor Y)$
8. $(B \equiv C) \supset (A \supset X)$
9. $(A \bullet Z) \lor {\sim}(X \bullet C)$
10. $(Z \bullet A) \lor ({\sim}C \bullet Y)$
11. $X \bullet [A \supset (Y \lor Z)]$
12. $(B \lor X) \supset {\sim}(Y \equiv C)$
13. $({\sim}B \supset Z) \bullet (A \equiv X)$
14. ${\sim}(A \equiv C) \supset (X \bullet Y)$
15. ${\sim}(A \lor Z) \equiv (X \bullet Y)$
16. $(C \supset Y) \lor [(A \bullet B) \supset {\sim}X]$
17. $[(C \bullet Y) \lor Z] \equiv [{\sim}B \lor (X \supset Y)]$
18. $[(X \bullet A) \supset B] \equiv [C \lor {\sim}(Z \supset Y)]$
19. $[(A \bullet B) \equiv X] \supset [({\sim}Z \bullet C) \lor Y]$
20. $[X \supset (A \lor B)] \equiv [(X \bullet Y) \lor (Z \bullet C)]$

Exercises 1.5b. Assume A, B, C are true; X, Y, Z are false; and P and Q are unknown. Evaluate the truth value of each complex expression.

1. $Q \bullet {\sim}Q$
2. $Q \supset B$
3. $P \bullet {\sim}C$
4. $P \equiv {\sim}P$
5. $P \vee (X \bullet Y)$
6. ${\sim}(Z \bullet A) \supset P$
7. $Q \vee {\sim}(Z \bullet A)$
8. $(P \supset A) \bullet (Z \vee B)$
9. $(Q \bullet C) \supset (X \vee A)$
10. ${\sim}\{[(Y \bullet B) \vee Z] \supset P\}$
11. $(C \vee X) \supset (Q \vee A)$
12. $[Q \bullet (B \equiv C)] \bullet {\sim}Y$
13. $({\sim}Z \supset X) \bullet (P \bullet {\sim}B)$
14. ${\sim}(Q \supset C) \vee (Z \bullet {\sim}X)$
15. $[(A \vee X) \supset (Y \bullet B)] \equiv {\sim}Q$
16. ${\sim}(A \vee P) \equiv [(B \bullet X) \supset Y]$
17. ${\sim}P \supset [{\sim}(A \bullet B) \vee (Z \bullet Y)]$
18. $(Q \bullet Z) \supset {\sim}[A \vee (X \equiv C)]$
19. $(Q \bullet X) \equiv [(A \vee {\sim}Z) \supset Y]$
20. $[(P \bullet Y) \vee {\sim}B] \equiv \{{\sim}A \supset [(C \vee X) \bullet Z]\}$

Exercises 1.5c. As in Exercises 1.5b, assume A, B, C are true; X, Y, Z are false; and P and Q are unknown. Evaluate the truth value of each complex expression.

1. $(P \bullet Q) \supset (X \vee A)$
2. $(Q \supset P) \bullet (Z \vee {\sim}Y)$
3. $(P \bullet Z) \supset (Q \vee A)$
4. $(P \vee Q) \vee ({\sim}A \equiv Y)$
5. ${\sim}\{[P \supset (Q \supset C)] \bullet Z\}$
6. $(Q \bullet P) \vee ({\sim}Q \vee {\sim}P)$
7. $[(Q \supset (P \supset Z)] \vee {\sim}({\sim}X \vee C)$
8. $\{Z \supset [P \supset (Q \supset A)]\} \supset (X \bullet Q)$
9. $[(Q \supset B) \bullet (X \vee {\sim}Z)] \equiv [P \supset (Q \supset {\sim}Y)]$
10. ${\sim}\{[(P \supset A) \bullet X] \equiv [(Q \vee {\sim}Q) \supset {\sim}B]\}$

§1.6. Truth Tables for Propositions

As we saw in §1.5, when we are given a complex proposition and we know the truth values of the component propositions, we can calculate the truth value of the longer statement. When we are given a complex proposition, and at least some of the truth values of the component propositions are unknown, sometimes we can determine the truth value of the complex proposition. But, sometimes the best we can do is to describe how the truth value of the whole varies with the truth value of its parts. We can construct truth tables for any proposition, with any number of component propositions of unknown truth values, using the basic truth tables. Truth tables summarize the distributions of all possible truth values of component propositions.

We can use truth tables to help us characterize complex propositions. Some complex propositions have interesting properties. Also, knowing the relations among the truth conditions of different complex propositions can also be useful. Most importantly, we can also use truth tables to separate valid from invalid arguments, the central task of this book.

We construct truth tables for wffs of **PL** in three steps.

Step 1. Determine how many rows we need.
Step 2. Assign truth values to the component variables.
Step 3. Work inside out, placing the column for each letter or connective directly beneath the letter or connective, until we complete the column under the main operator.

For Step 1, recall that the number of rows of a truth table is a function of the number of variables in the wff.

1 variable: 2 rows
2 variables: 4 rows
3 variables: 8 rows
4 variables: 16 rows
n variables: 2^n rows

For Step 2, it is conventional to start truth tables in a systematic way. For an example of a two-row truth table, consider the truth table for ‘P ⊃ P’.

P	⊃	P
1	**1**	1
0	**1**	0

For an example of a four-row truth table, consider 1.6.1

1.6.1 (P ∨ ~Q) • (Q ⊃ P)’

(P	∨	~	Q)	•	(Q	⊃	P)
1			1		1		1
1			0		0		1
0			1		1		0
0			0		0		0

Note that we use the same values we assign to P in the first column for P in the last column, and similarly for Q. All four row truth tables begin with this set of assignments. Let's see how to continue this example, in stages. First complete the columns under the tilde.

(P	∨	~	Q)	•	(Q	⊃	P)
1		**0**	1		1		1
1		**1**	0		0		1
0		**0**	1		1		0
0		**1**	0		0		0

Then we can complete the columns under the disjunction and the conditional.

(P	∨	~	Q)	•	(Q	⊃	P)
1	**1**	0	1		1	**1**	1
1	**1**	1	0		0	**1**	1
0	**0**	0	1		1	**0**	0
0	**1**	1	0		0	**1**	0

Finally, we can complete the truth table by completing the column under the main operator, the conjunction, using the columns for the disjunction and the conditional:

(P	∨	~	Q)	•	(Q	⊃	P)
1	**1**	0	1	**1**	1	**1**	1
1	**1**	1	0	**1**	0	**1**	1
0	**0**	0	1	**0**	1	**0**	0
0	**1**	1	0	**1**	0	**1**	0

Thus, 1.6.1 is false when P is false and Q is true, and true otherwise. Note that you only have to write out the truth table once, like the last one in this demonstration.

Here is the start to an eight-line truth table, for 1.6.2, which we will complete later.

1.6.2 $[(P \supset Q) \bullet (Q \supset R)] \supset (P \supset R)$

[(P	⊃	Q)	•	(Q	⊃	R)]	⊃	(P	⊃	R)
1		1		1		1		1		1
1		1		1		0		1		0
1		0		0		1		1		1
1		0		0		0		1		0
0		1		1		1		0		1
0		1		1		0		0		0
0		0		0		1		0		1
0		0		0		0		0		0

In general, to construct a truth table:

Method for Constructing Truth Tables

The first variable (reading from left to right, is assigned 1 in the top half and assigned 0 in the bottom half.
The second variable is assigned 1 in the top quarter, 0 in the second quarter, 1 in the third quarter, and 0 in the bottom quarter.
The third variable is assigned 1 in the top eighth, 0 in the second eighth...
...
The last variable is assigned alternating instances of 1 and 0.

Thus, in a 128 row truth table (7 variables), the first variable would get 64 1s and 64 0s, the second variable would get 32 1s, 32 0s, 32 1s, and 32 0s, the third variable would alternate 1s and 0s in groups of 16, the fourth variable would alternate 1s and 0s in groups of 8s... and the seventh variable would alternate single instances of 1s and 0s. It does not matter which variables we take as first, second, third, etc., but it is conventional that we work from left to right. Remember that every instance of the same variable letter gets the same assignment of truth values.

The technical work of constructing truth tables for propositions of any length allows us to classify individual propositions and their relations in a variety of interesting ways. For individual propositions, we can use truth tables to characterize the difference between logical necessity and logical contingency. Necessity and contingency are complicated concepts, so let’s take a moment to characterize them.

Necessity is easily defined in terms of possibility: a proposition is necessary if it is not possible for it to be false. A proposition is possible if it is not necessarily false. A proposition is contingent if it is possible, but not necessary.

To get a sense of what we mean by these terms, consider 1.6.3 - 1.6.5.

1.6.3	Aristotle distinguished four kinds of causes.
1.6.4	Descartes defended mind-body materialism.
1.6.5	$2 + 2 = 4$
1.6.6	$2 + 2 = 5$

1.6.3 and 1.6.5 are true; 1.6.4 and 1.6.6 are false. But, 1.6.5 is often taken to be necessarily true whereas 1.6.3 is usually seen as merely contingently true. Similarly, 1.6.5 is usually called contingently false whereas 1.6.6 may be taken as necessarily false.

The concepts of necessity and possibility, though clearly inter-related, are philosophically contentious. The claim that a proposition is necessary is often understood as the claim that it is true in all possible worlds, but that doesn't help very much. It is difficult to know how we acquire and justify beliefs about other possible worlds. Moreover, philosophers distinguish a variety of different kinds of necessity, among them are different kinds of logical necessity, metaphysical necessity, and physical necessity.

Fortunately, there is a kind of logical necessity which avoids most philosophical concerns about our access to possible worlds and which we can use truth tables to characterize. We can use truth tables to make distinctions among tautologies, contingencies, and contradictions. Consider again the truth table for 'P ⊃ P'.

P	⊃	P
1	**1**	1
0	**1**	0

'P ⊃ P' is a *tautology*, or a statement that is true in every row of the truth table. Two common laws of logic describe tautologies.

1.6.7 The Law of the Excluded Middle: any statement of the form '$\alpha \vee \sim\alpha$' is a tautology.

1.6.8 The Law of Non-Contradiction: any statement of the form '$\sim(\alpha \bullet \sim\alpha)$' is a tautology.

Tautologies are the theorems of propositional logic. They are sometimes called logical truths. 1.6.9 is a tautology in English.

1.6.9 Either the Phillies will win the World Series this year, or they will not.

Not all necessary truths are tautologies. 1.6.5 is not a logical truth, even if it is necessarily true. For most, perhaps all, necessary truths, there are some philosophers who claim that they are really contingent. But the logical truths are among the least controversial. All tautologies are logically necessary. 1.6.10 is a longer tautology

1.6.10 $[(P \supset Q) \bullet (Q \supset R)] \supset (P \supset R)$

[(P	⊃	Q)	•	(Q	⊃	R)]	⊃	(P	⊃	R)
1	1	1	1	1	1	1	**1**	1	1	1
1	1	1	0	1	0	0	**1**	1	0	0
1	0	0	0	0	1	1	**1**	1	1	1
1	0	0	0	0	1	0	**1**	1	0	0
0	1	1	1	1	1	1	**1**	0	1	1
0	1	1	0	1	0	0	**1**	0	1	0
0	1	0	1	0	1	1	**1**	0	1	1
0	1	0	1	0	1	0	**1**	0	1	0

Only a small portion of the sentences of propositional logic are tautologies. 1.6.11 is a *contingency*, or a statement that may or may not be true. It is true in at least one row of the truth table. It is false in at least one row. The truth value of a contingency depends on the values of its component premises.

1.6.11 $P \vee \sim Q$

P	∨	~	Q
1	**1**	0	1
1	**1**	1	0
0	**0**	0	1
0	**1**	1	0

While most wffs are contingent, some are false in every row. We call such statements *contradictions*. 1.6.12 and 1.6.13 are contradictions.

1.6.12 $P \bullet \sim P$

P	•	~	P
1	0	0	1
0	0	1	0

1.6.13 $(\sim P \supset Q) \equiv \sim(Q \vee P)$

(~	P	⊃	Q)	≡	~	(Q	∨	P)
0	1	1	1	**0**	0	1	1	1
0	1	1	0	**0**	0	0	1	1
1	0	1	1	**0**	0	1	1	0
1	0	0	0	**0**	1	0	0	0

In addition to helping us characterize individual propositions, truth tables give us tools to characterize relations among two or more propositions. Propositions can have the same values or opposite values. Consider the tautology 1.6.14.

1.6.14 $(A \vee B) \equiv (\sim B \supset A)$

(A	∨	B)	≡	(~	B	⊃	A)
1	1	1	**1**	0	1	1	1
1	1	0	**1**	1	0	1	1
0	1	1	**1**	0	1	1	0
0	0	0	**1**	1	0	0	0

We can eliminate the biconditional and consider the two remaining portions, the left side and the right side, as separate statements. The resulting propositions, 1.6.15 and 1.6.16, are *logically equivalent*: two or more statements with identical truth values in every row of the truth table.

1.6.15 $A \vee B$
1.6.16 $\sim B \supset A$

A	∨	B		~	B	⊃	A
1	**1**	1		0	1	**1**	1
1	**1**	0		1	0	**1**	1
0	**1**	1		0	1	**1**	0
0	**0**	0		1	0	**0**	0

The concept of logical equivalence has many uses. For one, notice that the biconditional is a superfluous connective, since any statement made with the biconditional could be made, in slightly more complex form, with a conjunction of two conditionals. That is, a statement of the form '$\alpha \equiv \beta$' is logically equivalent to a statement which uses only other connectives, a statement of the form '$(\alpha \supset \beta) \bullet (\beta \supset \alpha)$'.

α	≡	β		(α	⊃	β)	•	(β	⊃	α)
1	**1**	1		1	1	1	**1**	1	1	1
1	**0**	0		1	0	0	**0**	0	1	1
0	**0**	1		0	1	1	**0**	1	0	0
0	**1**	0		0	1	0	**1**	0	1	0

Other connectives can be shown to be superfluous, in similar ways. When constructing languages for propositional logic, we have choices of which connectives to use and how many connectives to use. The study of the relations among the different connectives is a topic in metalogic which I discuss in Chapter 4, §5.

In contrast to logically equivalent statements, 1.6.17 and 1.6.18 are contradictory: two statements with opposite truth values in all rows of the truth table.

1.6.17 A ∨ ~B
1.6.18 B • ~A

A	∨	~	B		B	•	~	A
1	**1**	0	1		1	**0**	0	1
1	**1**	1	0		0	**0**	0	1
0	**0**	0	1		1	**1**	1	0
0	**1**	1	0		0	**0**	1	0

Notice that the biconditional connecting the two contradictory statements is a contradiction. Most pairs of statements, like 1.6.19 and 1.6.20, are neither logically equivalent nor contradictory.

1.6.19 E ⊃ D
1.6.20 ~E • D

E	⊃	D		~	E	•	D
1	**1**	1		0	1	**0**	1
1	**0**	0		0	1	**0**	0
0	**1**	1		1	0	**1**	1
0	**0**	0		1	0	**0**	0

We can see that 1.6.19 and 1.6.20 are not contradictory in rows 2, 3, and 4. We can see that they are not logically equivalent in row 1. But, an interesting property of this pair of statements is that they are consistent: there are values of the component variables that will make both propositions true. 1.6.19 and 1.6.20 are both true in row 3. A person who asserts propositions of both forms can be making consistent statements. It depends on the interpretations of the variables E and D. If two statements are neither logically equivalent nor contradictory, they may be consistent or inconsistent.

1.6.21 *Consistent* propositions can be true together. There is at least one row of the truth table in which two or more propositions are all true.

1.6.22 *Inconsistent* pairs of propositions are not consistent. There is no row of the truth table in which both statements are true.

1.6.23 and 1.6.24 are an inconsistent pair.

1.6.23 $E \bullet F$

1.6.24 $\sim(E \supset F)$

E	•	F		~	(E	⊃	F)
1	**1**	1		**0**	1	1	1
1	**0**	0		**1**	1	0	0
0	**0**	1		**0**	0	1	1
0	**0**	0		**0**	0	1	0

Notice that the conjunction of two inconsistent statements is a self-contradiction. The difference between two sentences which are inconsistent and two sentences which are contradictory is subtle. In both cases, the pair of sentences can not be true together. The difference is whether the pair can be false in the same conditions. Contradictory pairs always have opposite truth values. Inconsistent pairs may have truth conditions in which they are both false. When we are making assertions, and aiming at the truth, it is generally just as bad to make inconsistent assertions as it is to make contradictory assertions.

When comparing two propositions, first look for the stronger conditions: logical equivalence and contradiction. Then, if these fail, look for the weaker conditions: consistency and inconsistency.

Exercises 1.6a. Construct truth tables for each of the following propositions.

1. A ⊃ ~A
2. B ⊃ (~B ⊃ B)
3. (C • ~C) ⊃ C
4. (D ∨ ~D) ≡ D
5. ~E ⊃ F
6. G ≡ ~H
7. I • (J ∨ I)
8. (K ≡ L) ⊃ L
9. (M • N) ∨ ~M
10. ~O ∨ (P ⊃ O)
11. (Q ⊃ R) ≡ (R ⊃ Q)
12. (S ∨ T) • ~(T ⊃ S)
13. (U • ~V) ⊃ (V ∨ U)
14. ~[(W ∨ X) • ~X] ⊃ W
15. [(~Y • Z) ⊃ Y] ∨ (Y ≡Z)
16. (A ≡ ~B) ⊃ [(B ∨ ~B) • A]
17. (C ∨ D) ⊃ E
18. (F • G) ≡ H
19. ~(I ∨ J) • K
20. [L ⊃ (M ∨ N)] ≡ L
21. [~O • (P ⊃O)] ∨ Q
22. (~R ∨ S) • (~T ⊃ R)
23. [U ⊃ (V ⊃ W)] • (V ∨ W)
24. [~X ≡ (Y • Z)] ⊃ (X ∨ Z)
25. (A ⊃ B) ∨ (C ≡ D)
26. (E • ~F) ⊃ (G ∨ H)
27. [I ⊃ (J • K)] ∨ (L ≡ I)
28. [(~M • N) ∨ (O ⊃ P)] ≡ M

Exercises 1.6b. Classify each proposition as tautologous, contingent, or contradictory.

1. A ∨ ~A
2. B ≡ ~B
3. ~C ⊃ ~C
4. (D ⊃ ~D) ∨ D
5. E ≡ (~E ⊃ E)
6. (~F ⊃ F) • F
7. (G • ~G) ⊃ (G ∨ ~G)
8. B ≡ (A • ~B)
9. (C ∨ D) • ~(D ⊃ C)
10. (E ⊃ F) ≡ ~(F ∨ ~E)
11. (G • ~H) ∨ (H ⊃ ~G)
12. ~(I • J) ≡ (~I ∨ ~J)
13. (K ⊃ L) ≡ (K • ~L)
14. (~M • N) • (N ⊃ M)
15. (~P ≡ Q) • ~[Q ⊃ (P ∨ Q)]
16. (R ∨ S) ≡ (~R • ~S)
17. (T ⊃ U) ∨ (U ⊃ T)
18. ~C ≡ (A ∨ ~B)
19. (D ⊃ F) ∨ (E ⊃ D)
20. (G • H) ⊃ (G ∨ I)
21. (J • ~K) • ~(L ∨ J)
22. (N ∨ O) ⊃ (M • O)
23. ~(P • Q) ∨ (Q ⊃ R)
24. (T ≡ S) ⊃ [~U • (S • T)]
25. ~{[(X • Y) ⊃ Z] ≡ [X ⊃ (Y ⊃ Z)]}
26. [~A ∨ (~B • ~C)] ≡ [(A • B) ∨ (A • C)]
27. [D ∨ (E • F)] ≡ [(D ∨ E) • (D ∨ F)]
28. (G ∨ H) ∨ (I ∨ J)
29. [K • (L ⊃ M)] ∨ (N ≡ K)
30. [P ⊃ (Q • R)] ⊃ [~S ≡ (P ∨ R)]

Exercises 1.6c. Are the following pairs of statements logically equivalent or contradictory? If neither, are they consistent or inconsistent?

1. A ⊃ B	and	B ⊃ A
2. ~(C • D)	and	C ∨ D
3. ~E ⊃ ~F	and	E ∨ F
4. G ⊃ H	and	~H • G
5. I ∨ J	and	~I • ~J
6. K ≡ L	and	~(L ⊃ K)
7. ~(M ∨ N)	and	~M • ~N
8. ~O ⊃ P	and	O ∨ P
9. ~Q ≡ R	and	Q • R
10. (S ∨ T) • ~S	and	T ⊃ S
11. (U ≡ X) ∨ ~U	and	~X ⊃ ~U
12. ~Y ⊃ Z	and	~Z ⊃ Y
13. ~(A • B)	and	~A ⊃ B
14. C ⊃ (D • C)	and	~D • C
15. (E ∨ F) • E	and	~ (E ∨ F)
16. (G • H) ∨ ~G	and	~H ⊃ (G ≡ H)
17. I ∨ (J • ~J)	and	(J ≡ ~I) • J
18. K • (L ⊃ K)	and	(K ∨ L) ≡ ~K
19. (~M • ~N) ≡ N	and	(N ∨ M) • ~M
20. (O ∨ P) ⊃ O	and	~O ≡ (P • O)
21. (Q ∨ R) • S	and	(Q ⊃ S) • R
22. T ∨ (U • W)	and	(T ∨ U) • (T ∨ W)
23. (X • Y) ∨ Z	and	(~X ∨ ~Y) • ~Z
24. (A • B) ⊃ C	and	A ⊃ (B ⊃ C)
25. (D ≡ E) ⊃ F	and	(D ∨ F) • ~E
26. ~(G ∨ H) • I	and	(I ⊃ G) • H
27. (J ≡ K) • L	and	[(~L ∨ ~K) • (L ∨ K)] ∨ ~L
28. (M ⊃ N) ∨ (N • ~O)	and	(M • ~N) • (~N ∨ O)
29. P ∨ (Q ∨ R)	and	(P ∨ Q) ∨ R
30. S ⊃ (T ≡ U)	and	[(T ∨ U) • S] ⊃ ~T
31. (X • Y) ⊃ Z	and	(X • Y) • ~Z
32. (A ⊃ B) • C	and	(~B ⊃ ~A) • C
33. (~D ⊃ ~E) ∨ (F ≡ E)	and	(~D • E) • [(~F ∨ ~E) • (F ∨ E)]
34. [(G ∨ H) ⊃ I] • [~I ∨ H]	and	[(G ∨ H) • ~I] ∨ (I • ~H)
35. (~K ⊃ L) • ~M	and	M ≡ (L ∨ K)

§1.7. Valid and Invalid Arguments

The central task of this book is to characterize a notion of logical consequence, of which arguments are valid. We can use truth tables to define validity in **PL**. Contrast the arguments 1.7.1 and 1.7.2.

1.7.1
1. If God exists then every effect has a cause.
2. God exists.
So, every effect has a cause.

1.7.2 .
1. If God exists then every effect has a cause.
2. Every effect has a cause.
So, God exists.

1.7.1 is valid. There is no way for the premises to be true while the conclusion is false. Whether the premises are true, whether 1.71 is a sound argument, is a separate question. 1.7.1 has the form 1.7.3.

1.7.3
$\alpha \supset \beta$
$\alpha \quad / \beta$

1.7.3 is known as Modus Ponens. Note that we write the premises on sequential lines, and the conclusion on the same line as the final premise, following a single slash. In 1.7.3, we are using metalinguistic variables (Greek letters) to indicate that any consistent substitution of those variables by symbols of **PL** is a valid argument.

1.7.2 is invalid, and has the form 1.7.4.

1.7.4
$\alpha \supset \beta$
$\beta \quad / \alpha$

Arguments of the form 1.7.4 commit the fallacy of affirming the consequent. The conclusion of 1.7.2 fails to follow from the premises. It is logically possible for its premises to be true while its conclusion is false. This fallacy is a formal result having nothing to do with the content of the propositions used in the argument.

We need a rigorous method for distinguishing valid argument forms like 1.7.3 from invalid ones like 1.7.4. The truth table method for determining if an argument is valid is simple.

Method of Truth Tables to Test for Validity

Step 1: Line up premises and conclusion horizontally, separating premises with a single slash and separating the premises from the conclusion with a double slash.
Step 2: Construct truth tables for each premise and the conclusion, using consistent assignments to component variables.
Step 3: Look for a counterexample: a row in which all premises are true and the conclusion is false.
If there is a counterexample, the argument is invalid. Specify a counterexample.
If there is no counterexample, the argument is valid.

Recall that in a *valid argument*, if the premises are true then the conclusion must be true. This definition says nothing about what happens if the premises are false. An *invalid argument* is one in which it is possible for true premises to yield a false conclusion. By focusing on valid arguments, we can make sure that if all our premises are true, our conclusions be true as well.

Let's examine the argument 1.7.5 to determine whether it is valid.

1.7.5 $\quad P \supset Q$
$\quad\quad\quad P \quad / Q$

P	⊃	Q	/	P	//	Q
1	**1**	1		1		1
1	**0**	0		1		0
0	**1**	1		0		1
0	**1**	0		0		0

Notice that in no row are the premises true and the conclusion false. There is no counterexample. 1.7.5 is a valid argument. In contrast, both 1.7.6 and 1.7.7 are invalid arguments. To show that they are invalid, we specify a counterexample.

1.7.6 $\quad P \supset Q$
$\quad\quad\quad Q \quad / P$

P	⊃	Q	/	Q	//	P
1	**1**	1		1		1
1	**0**	0		0		1
0	**1**	1		1		0
0	**1**	0		0		0

The third row of 1.7.6 is a counterexample. The argument is shown invalid when P is false and Q is true. Some arguments will have more than one counterexample; demonstrating one counterexample is sufficient to show that an argument is invalid.

1.7.7 $\quad P \supset (Q \supset P)$
$\quad\quad\quad {\sim}P \quad / Q$

P	⊃	(Q	⊃	P)	/	~	P	//	Q
1	**1**	1	1	1		**0**	1		**1**
1	**1**	0	1	1		**0**	1		**0**
0	**1**	1	0	0		**1**	0		**1**
0	**1**	0	1	0		**1**	0		**0**

In 1.7.7, row 4 is a counterexample. The argument is shown invalid when P is false and Q is false.

Exercises 1.7. Construct truth tables to determine whether each argument is valid. If an argument is invalid, specify a counterexample.

1. $A \supset {\sim}A$
$\quad {\sim}A \qquad / \; A$

2. $\quad {\sim}{\sim}B \vee ({\sim}B \supset B) \qquad / \; B$

3. $C \vee D$
$\quad {\sim}D \qquad / \; {\sim}C$

4. $E \vee F$
$\quad {\sim}(E \cdot {\sim}F) \qquad / \; E \equiv F$

5. $G \equiv H$
$\quad {\sim}H \qquad / \; {\sim}G$

6. ${\sim}I \supset ({\sim}I \supset I)$
$\quad {\sim}J \supset ({\sim}J \supset J) \qquad / \; I \cdot J$

7. $(K \cdot L) \vee (K \cdot {\sim}L)$
$\quad {\sim}K \qquad / \; L$

8. $M \equiv {\sim}N$
$\quad M \vee N$
$\quad M \qquad / \; {\sim}N \supset N$

9. ${\sim}P \supset Q$
$\quad Q \supset P \qquad / \; {\sim}P$

10. $R \supset S$
$\quad S \vee T \qquad / \; R \supset T$

11. $X \cdot {\sim}Y$
$\quad Y \vee Z \qquad / \; {\sim}Z$

12. ${\sim}(A \cdot B)$
$\quad B \supset C \qquad / \; A$

13. $D \vee E$
$\quad {\sim}D \cdot {\sim}F \qquad / \; {\sim}(E \cdot F)$

14. $G \equiv H$
$\quad H \cdot {\sim}I \qquad / \; {\sim}(I \cdot G)$

15. $J \supset {\sim}K$
$\quad K \supset L \qquad / \; {\sim}(L \cdot J)$

16. $M \cdot {\sim}N$
$\quad O \supset P$
$\quad P \vee N \qquad / \; {\sim}M$

17. $Q \supset R$
$\quad S \vee T$
$\quad T \qquad / \; R$

18. ${\sim}W \supset (X \vee Y)$
$\quad Y \cdot Z$
$\quad {\sim}(Z \supset X) \qquad / \; W \equiv Y$

19. ${\sim}A \cdot (B \vee C)$
$\quad C \supset A$
$\quad B \supset D \qquad / \; A \supset {\sim}D$

20. $E \cdot F$
$\quad G \supset (H \vee {\sim}E)$
$\quad {\sim}F \vee G \qquad / \; H$

§1.8. Indirect Truth Tables

We can use the truth table method for any argument, to determine its validity. This gives us a mechanical procedure for determining counterexamples. But the method becomes unwieldy as the number of variables in an argument grows. With merely five variables, for example, a truth table is 32 lines. The truth table for an argument which contains ten propositional variables would require 1024 lines.

Fortunately, there is a shortcut method for constructing counterexamples. This method will also help for determining whether a set of propositions is consistent.

To show that an argument is valid, you have to show that there is no row of the truth table with true premises and a false conclusion. We have to examine every row. But we only need one row in order to demonstrate that an argument is invalid: a counterexample. Thus, to determine whether an argument is valid or invalid, we can try to construct a counterexample. If we find a counterexample, then we know the argument is invalid. If there are no counterexamples, then the argument is valid.

Method of Indirect Truth Tables for Validity

Step 1. Assign values to make the conclusion false.
Step 2. Try to make all premises true.

If Steps 1 and 2 are possible, then the argument is invalid; specify the counterexample.
If Steps 1 and 2 are impossible, then the argument is valid.

Steps 1 and 2 may be completed in any order. If there is a counterexample, this indirect method will be able to find it. But, we have to make sure to try all ways of assigning values to the component propositions.

1.8.1 is an invalid argument. We will use the indirect method to find a counterexample.

1.8.1 $\quad G \equiv H$
$\quad\quad\quad G \quad / \sim H$

To show that 1.8.1 is invalid, first write it out, as you would a normal truth table for an argument.

G	≡	H	/	G	//	~	H

Next, we can assign the value true to H, in order to make the conclusion false.

G	≡	H	/	G	//	~	H
						0	1

Carry this value over to any other H in the argument.

G	≡	H	/	G	//	~	H
		1				0	1

Assign a value to G which makes the premises true.

G	≡	H	/	G	//	~	H
1	1	1		1		0	1

1.8.1 is thus invalid since there is a counterexample when G is true and H is true. Note that an argument is either valid or invalid. If there is at least one counterexample, the argument is invalid. It is not merely invalid on that assignment of truth values; it is always invalid.

1.8.2 is a valid argument. We will not be able to construct a counterexample.

1.8.2 $\quad$ C ⊃ (D ⊃ E)
D ⊃ (E ⊃ F) $\quad$ / C ⊃ (D ⊃ F)

The only way to make the conclusion false is to assign true to C and to D, and false to F.

C	⊃	(D	⊃	E)	/	D	⊃	(E	⊃	F)	//	C	⊃	(D	⊃	F)
												1	0	1	0	0

Carry these values over to the premises.

C	⊃	(D	⊃	E)	/	D	⊃	(E	⊃	F)	//	C	⊃	(D	⊃	F)
1		1				1				0		1	0	1	0	0

In order to make the first premise true, E must also be true.

C	⊃	(D	⊃	E)	/	D	⊃	(E	⊃	F)	//	C	⊃	(D	⊃	F)
1	1	1	1	1		1		1		0		1	0	1	0	0

But now the second premise is false. If we tried to make the second premise true, by making E false, the first premise would come out false. There was no other way to make the conclusion false. So there is no counterexample. 1.8.2 is thus valid.

In some arguments, there is more than one way to make a conclusion false or to make premises true. You may have to try more than one. Once you arrive at a counterexample, you may stop. But, if you fail to find a counterexample, you must keep going until you have tried all possible assignments.

Some arguments. like 1.8.3, have multiple counterexamples. Remember, you only need one to demonstrate the invalidity of an argument.

1.8.3 $\quad$ I ⊃ K
K ⊃ J $\quad$ / I • J

There are three ways to make the conclusion of 1.8.3 false.

I	⊃	K	/	K	⊃	J	//	I	•	J
								1	0	0
								0	0	1
								0	0	0

We can try them in order. On the first assignment, there is no way to assign a truth value to K which makes the premises true.

I	⊃	K	/	K	⊃	J	//	I	•	J
1		?		?		0		1	0	0
								0	0	1
								0	0	0

We must move on to the second option.

I	⊃	K	/	K	⊃	J	//	I	•	J
								1	0	0
0						1		0	0	1
								0	0	0

Here, we can assign either value to K and find a counterexample. 1.8.3 is shown invalid when I is false, J is true, and K is either true or false. Since we found a counterexample in the second option, there is no need to continue with the third option.

1.8.4 requires more work.

1.8.4 $\quad$ T ⊃ (U ∨ X)
U ⊃ (Y ∨ Z)
Z ⊃ A
~(A ∨ Y) $\quad$ / ~T

Start with the conclusion, making T true in order to make its negation false, carrying that assignment into the first premise.

T	⊃	(U	∨	X)	/	U	⊃	(Y	∨	Z)	/	Z	⊃	A	/	~	(A	∨	Y)	//	~	T
1																					0	1

From the first premise,‘U ∨ X’ must be true, but there are three ways to assign values to make it so. Similarly, there are multiple ways to assign values for the second and third premises. But in the fourth premise, there is only one possible assignment

T	⊃	(U	∨	X)	/	U	⊃	(Y	∨	Z)	/	Z	⊃	A	/	~	(A	∨	Y)	//	~	T
1																1	0	0	0		0	1

Let’s carry these assignments over.

T	⊃	(U	∨	X)	/	U	⊃	(Y	∨	Z)	/	Z	⊃	A	/	~	(A	∨	Y)	//	~	T
1								0						0		1	0	0	0		0	1

We know by inspecting the third premise that Z must also be false.

T	⊃	(U	∨	X)	/	U	⊃	(Y	∨	Z)	/	Z	⊃	A	/	~	(A	∨	Y)	//	~	T
1								0				0	1	0		1	0	0	0		0	1

Carrying this assignment over, ‘Y ∨ Z’ has been made false, and so U must be made false, also.

T	⊃	(U	∨	X)	/	U	⊃	(Y	∨	Z)	/	Z	⊃	A	/	~	(A	∨	Y)	//	~	T
1						0	1	0	0	0		0	1	0		1	0	0	0		0	1

We can carry the value of U to the first premise.

T	⊃	(U	∨	X)	/	U	⊃	(Y	∨	Z)	/	Z	⊃	A	/	~	(A	∨	Y)	//	~	T
1		0				0	1	0	0	0		0	1	0		1	0	0	0		0	1

We now are constrained to make X true, in order to make the first premise true.

T	⊃	(U	∨	X)	/	U	⊃	(Y	∨	Z)	/	Z	⊃	A	/	~	(A	∨	Y)	//	~	T
1	1	0	1	1		0	1	0	0	0		0	1	0		1	0	0	0		0	1

The counterexample is thus constructed. The argument is shown invalid when T and X are true; and U, Y, Z, and A are all false.

Using indirect truth tables to determine whether a set of propositions is consistent

The most important use of the indirect truth table method is in determining whether an argument is valid. An argument is valid if there is no assignment of truth values to the component propositional variables such that the premises come out true and the conclusion comes out false. Given our bivalent logic, that condition for validity is the same as testing whether the negation of the conclusion is consistent with the premises. For, a set of propositions is consistent when there is a set of truth values which we can assign to the component variables such that all the propositions come out true. In other words, we can use the same method for determining whether a set of propositions is consistent as we used for determining whether an argument is valid.

Two or more propositions are consistent if there is at least one line of the truth table in which they are all true. If we can find an assignment of truth values to component propositions which makes all the main operators of the propositions in a set true, then we have shown them consistent. This assignment will be called a consistent valuation. If no consistent valuation is possible, then the set is inconsistent.

Method of Indirect Truth Tables for Consistency

Assign values to propositional variables to make each statement true.
If you can make each statement true, then the set is consistent. Provide a consistent valuation.
If it is not possible to make each statement true, then the set is inconsistent.

To determine if a set of propositions is consistent, line them up, just as we lined up the premises and conclusion in evaluating arguments. We only use single slashes between the propositions. Since a bare set of sentences has no conclusion, the order makes no difference. We are trying to find a valuation which makes all propositions true. Let’s examine the set of propositions 1.8.5 to see if they are consistent.

1.8.5 (A • B) ⊃ F
B ⊃ (D ∨ ~E)
F ⊃ (E ∨ ~D)
A • E

(A	•	B)	⊃	F	/	B	⊃	(D	∨	~	E)	/	F	⊃	(E	∨	~	D)	/	A	•	E

Let’s start with the last proposition, since there is only one way to make it true

(A	•	B)	⊃	F	/	B	⊃	(D	∨	~	E)	/	F	⊃	(E	∨	~	D)	/	A	•	E
																				1	**1**	1

Carry those values through the rest of the set, and evaluate the negation in the second proposition.

(A	•	B)	⊃	F	/	B	⊃	(D	∨	~	E)	/	F	⊃	(E	∨	~	D)	/	A	•	E
1										0	1				1					1	**1**	1

There are no other obvious, forced moves. The antecedent in the first proposition may be either true or false, depending on the value given to B. The consequent in the second may be either true or false, depending on the value given to D. The consequent in the third proposition, and thus the whole third proposition, will be true, but that does not determine the values of either F or D.

(A	•	B)	⊃	F	/	B	⊃	(D	∨	~	E)	/	F	⊃	(E	∨	~	D)	/	A	•	E
1										0	1			**1**	1	1				1	**1**	1

We must arbitrarily choose a next place to work. I'll choose to start with B; we may assign either a 1 or a 0.

(A	•	B)	⊃	F	/	B	⊃	(D	∨	~	E)	/	F	⊃	(E	∨	~	D)	/	A	•	E
1		1				1				0	1			**1**	1	1				1	**1**	1
1		0				0				0	1			**1**	1	1				1	**1**	1

We'll try the first line, first. Assigning 1 to B forces us, in the first proposition to assign 1 to F.

(A	•	B)	⊃	F	/	B	⊃	(D	∨	~	E)	/	F	⊃	(E	∨	~	D)	/	A	•	E
1	1	1	**1**	1		1				0	1		1	**1**	1	1				1	**1**	1
1		0				0				0	1			**1**	1	1				1	**1**	1

Then, in the second proposition, we must assign 1 to D.

(A	•	B)	⊃	F	/	B	⊃	(D	∨	~	E)	/	F	⊃	(E	∨	~	D)	/	A	•	E
1	1	1	**1**	1		1	**1**	1	1	0	1		1	**1**	1	1	0	1		1	**1**	1
1		0				0				0	1			**1**	1	1				1	**1**	1

We have thus found a consistent valuation. We need not return to the problem to assign 0 to B; all we need is a single valuation to show the set consistent. The set of propositions is shown consistent when all component variables are assigned 1. There are other possible consistent valuations.

Just as an argument is invalid if there is at least one counterexample, a set of propositions is consistent if there is at least one consistent valuation. If there is no consistent valuation, the set is inconsistent.

We will use an extended version of this indirect truth table method for determining counterexamples to arguments again in Chapter 3, in first-order logic. For now, there are two salient applications of the method. When determining if an argument is valid, the method, if used properly, will generate a counterexample if there is one. For sets of sentences, the method will yield a consistent valuation. Make sure to work until you have exhausted all possible assignments of truth values to the simple, component propositions.

Exercises 1.8a. Determine whether each of the following arguments is valid. If invalid, specify a counterexample.

1. $A \supset (B \vee C)$
$C \bullet (\sim D \supset A)$
$E \bullet B \qquad / \; E \bullet A$

2. $F \equiv (G \vee H)$
$I \supset (J \supset F)$
$(I \bullet G) \vee H \qquad / \; J \supset G$

3. $M \bullet \sim(K \equiv O)$
$(L \vee N) \supset O$
$M \vee \sim K \qquad / \; L \vee O$

4. $P \supset \sim Q$
$Q \supset (S \bullet T)$
$\sim T \vee (P \supset R) \qquad / \; Q \supset R$

5. $(Z \bullet V) \supset (U \vee W)$
$X \vee (\sim Y \equiv W)$
$Z \bullet Y \qquad / \; \sim U$

6. $A \bullet B$
$B \supset C$
$\sim B \vee (D \supset \sim C) \qquad / \; \sim D$

7. $D \vee \sim E$
$(F \bullet G) \bullet \sim H$
$D \supset (H \vee I)$
$\sim I \qquad / \; F \bullet \sim E$

8. $J \supset (K \bullet \sim L)$
$\sim L \equiv (N \supset M)$
$J \vee \sim N$
$K \bullet M \qquad / \; \sim N$

9. $\sim(P \supset Q)$
$R \equiv (S \vee \sim T)$
$P \supset R$
$Q \vee T \qquad / \; S \vee O$

10. $(U \bullet \sim V) \vee W$
$W \equiv \sim V \qquad / \; \sim U$

11. $\sim Y \equiv (\sim X \bullet Z)$
$Z \supset Y \qquad / \; Z \supset X$

12. $B \bullet (D \vee C)$
$D \supset (A \vee E)$
$\sim E \vee (B \bullet C) \qquad / \; (A \supset E) \vee C$

13. $(F \vee G) \equiv (H \bullet J)$
$(I \supset H) \bullet (J \vee G)$
$\sim G \qquad / \; I \supset F$

14. $K \supset (M \supset P)$
$P \bullet \sim(N \vee L)$
$O \supset (K \equiv N) \qquad / \; M$

15. $Q \supset (T \bullet S)$
$R \equiv (U \vee T)$
$\sim[S \supset (T \supset Q)] \qquad / \; \sim U \bullet S$

16. $Y \supset (Z \equiv X)$
$Y \bullet \sim W$
$W \supset (Y \vee Z) \qquad / \; \sim(X \supset \sim W)$

17. $(B \supset A) \bullet E$
$\sim F \bullet (D \equiv C)$
$C \supset (E \bullet B)$
$\sim A \qquad / \; D$

18. $G \supset (H \vee K)$
$J \vee \sim H$
$I \vee (K \equiv J)$
$\sim I \qquad / \; H \vee G$

19. $L \supset (M \equiv \sim N)$
$(M \bullet O) \vee (\sim P \bullet O)$
$O \vee L$
$\sim M \qquad / \; N$

20. $R \supset [U \vee (S \vee Q)]$
$R \bullet \sim S$
$\sim U \equiv T \qquad / \; T \supset Q$

21. $V \supset (Z \bullet W)$
$X \vee \sim Y$
$Z \supset Y$
$V \equiv Y \qquad / \; \sim W$

22. $B \bullet (C \supset E)$
$B \supset (A \bullet F)$
$D \supset (\sim B \vee C)$
$E \supset D$ $\quad / A \equiv [F \bullet (E \supset C)]$

23. $G \supset (J \bullet \sim K)$
$(I \supset H) \bullet G$
$H \supset (K \vee I)$
$J \bullet \sim I$ $\quad / I \vee K$

24. $N \bullet (Q \supset P)$
$M \vee \sim L$
$L \supset Q$
$P \vee M$ $\quad / L \equiv M$

25. $U \bullet \sim R$
$(T \vee \sim S) \equiv U$
$R \vee S$
$T \supset (\sim R \vee V)$ $\quad / \sim V$

26. $E \bullet F$
$E \supset G$
$\sim G \bullet \sim H$ $\quad / F \equiv H$

27. $J \vee M$
$L \bullet M$
$K \supset L$ $\quad / \sim K \supset J$

28. $N \supset O$
$O \bullet P$
$P \equiv Q$ $\quad / \sim (Q \vee N)$

29. $T \equiv S$
$S \bullet U$
$R \supset U$ $\quad / R \vee T$

30. $Z \vee (X \bullet Y)$
$W \equiv V$
$Z \bullet V$ $\quad / W \supset (X \vee Y)$

31. $N \vee O$
$N \supset (Q \supset O)$
$(P \vee Q) \vee R$
$R \supset \sim R$ $\quad / \sim O \supset P$

32. $S \supset (V \bullet T)$
$U \vee R$
$\sim S \equiv (R \vee T)$ $\quad / T \supset U$

Exercises 1.8b. Determine, for each given set of propositions, whether it is consistent. If it is, provide a consistent valuation.

1. $A \vee B$
$B \bullet \sim C$
$\sim C \supset D$
$D \equiv A$

2. $B \bullet (C \supset A)$
$D \vee (E \bullet F)$
$F \supset (C \vee D)$
$E \bullet \sim A$

3. $D \supset F$
$F \equiv (A \bullet E)$
$D \bullet (B \vee C)$
$\sim A$
$E \vee C$

4. $F \bullet (A \supset D)$
$E \vee \sim B$
$\sim [C \supset (D \vee F)]$
$A \vee (B \bullet D)$
$E \supset A$

5. $\sim A \bullet \sim E$
$(A \vee B) \supset (D \bullet F)$
$C \supset (E \supset D)$
$\sim A \bullet (C \vee B)$

6. $\sim [A \supset (F \bullet B)]$
$B \bullet (E \bullet \sim D)$
$E \equiv F$
$D \supset (C \bullet A)$

7.
$C \equiv (D \vee B)$
$D \bullet (C \supset A)$
$\sim A \bullet (E \vee F)$
$F \supset (B \bullet A)$

8.
$D \supset (\sim A \bullet \sim F)$
$E \vee (\sim B \vee C)$
$E \supset C$
$A \bullet (\sim B \equiv D)$

9.
$B \vee (F \bullet D)$
$E \equiv B$
$\sim E \bullet \sim F$
$D \supset (A \supset C)$
$(C \bullet E) \vee A$

10.
$F \bullet (A \equiv \sim D)$
$(D \vee B) \supset E$
$(A \vee B) \equiv (D \supset F)$
$(F \bullet A) \supset \sim B$

11.
$(O \vee \sim P) \supset \sim Q$
$R \bullet (\sim S \vee T)$
$O \bullet \sim (R \supset Q)$
$P \supset S$

12.
$S \supset [O \bullet (\sim P \bullet R)]$
$S \vee (T \bullet \sim O)$
$R \supset (P \equiv T)$
$\sim S \vee R$

13.
$P \supset [Q \supset (R \supset O)]$
$\sim S \bullet T$
$R \equiv (T \bullet P)$
$\sim (O \vee S)$

14.
$\sim (T \bullet S) \bullet (\sim O \supset R)$
$S \bullet (O \vee \sim P)$
$R \supset (Q \bullet P)$
$T \vee \sim O$

15.
$T \supset (R \bullet \sim S)$
$(Q \vee R) \equiv (P \bullet T)$
$T \vee \sim Q$
$S \bullet (R \vee P)$

16.
$(\sim J \vee \sim K) \bullet L$
$\sim I \vee (M \vee N)$
$L \supset (I \bullet J)$
$(J \bullet M) \supset \sim N$

17.
$(L \bullet N) \vee I$
$L \equiv \sim K$
$K \supset (I \equiv \sim M)$
$(J \vee K) \bullet \sim N$

18.
$\sim (M \supset K)$
$(J \bullet L) \supset K$
$(J \vee M) \bullet (M \supset J)$
$K \vee L$

19.
$L \vee (K \bullet J)$
$J \supset (M \bullet N)$
$M \supset (I \vee J)$
$\sim [(N \supset K) \bullet L]$

20.
$\sim (J \supset N)$
$N \supset (M \bullet \sim L)$
$K \equiv \sim I$
$J \bullet (K \vee M)$
$I \bullet L$

Chapter 2: Inference in Propositional Logic
§1: Rules of Inference 1

We have used truth tables, including the indirect method, to separate valid arguments from invalid arguments. Our work was guided by a semantic definition of validity: an argument is valid if there are no assignments of truth values to the propositional variables on which the premises come out true and the conclusion comes out false. The full truth-table method gets increasingly and prohibitively arduous as the complexity of an argument grows. The indirect method, while pleasant and effective, requires ingenuity and can be just as laborious as the complete truth-table method. More importantly, while semantic tests of validity are effective in propositional logic, in more sophisticated logical systems they do not suffice to determine valid arguments. We should explore other methods.

This chapter explores one salient alternative method for determining valid inferences: the method of natural deduction. A natural deduction is sometimes called a proof or a derivation. Proofs, in a formal sense, are sequences of wffs every member of which either is an assumption or follows from earlier wffs in the sequence according to specified rules. These rules comprise a system of inference. Systems of inference are constructed first by specifying a language and then by adding rules governing derivations: how to get new wffs from the old ones. In addition, in some formal systems, some basic axioms are given. In formal systems of propositional logic, these axioms are ordinarily tautologies. Since tautologies are true in all cases, they can be added to any derivation. We will not use any axioms.

Just as we named our languages, we can also name our systems of inferences. Since we are mainly using only a single system of inference for each language, I won't bother to confuse us with more names than we need.

For natural deductions, our formal system will use the language of propositional logic, eight rules of inference and ten rules of equivalence. These rules are chosen so that they are complete: every valid argument will be provable using our rules. Our rules are also chosen arbitrarily, in the sense that there are many different complete systems of rules, indeed infinitely many. One can devise deductive systems with very few rules, though the resulting proofs become very long. One can devise systems so that proofs become very short, but the required number of rules can be unfeasibly large. We choose a moderate number of rules (eighteen) so that there are not too many to memorize and the proofs are not too long.

The rules we choose are defined purely syntactically, in terms of their form, but they are justified semantically. A rule of inference must preserve truth: given true premises, the rules must never yield a false conclusion. A rule preserves truth if every argument of its form is valid. We can prove that each of the rules of inference preserves truth using the indirect truth table method. We show that each rule of equivalence preserves truth using truth tables as well.

Derivations begin with any number of premises and proceed by steps to a conclusion. A derivation is valid if every step is either a premise or derived from premises or previous steps using our rules. I introduce four rules of inference in this section, the remaining four in the next section, and the ten rules of equivalence in the third and fourth sections of this chapter.

Let's start to examine our first rules. Observe that each of 2.1.1 - 2.1.3 are valid; we can use truth tables to show that they are valid.

2.1.1	$A \supset B$	
	A	$/ B$
2.1.2	$(E \bullet I) \supset D$	
	$(E \bullet I)$	$/ D$
2.1.3	$\sim G \supset (F \bullet H)$	
	$\sim G$	$/ F \bullet H$

2.1.1 - 2.1.3 share a common, valid form. We can write this form using metalinguistic variables; I will use Greek letters.

2.1.4	$\alpha \supset \beta$		
	α	/ β	Modus Ponens (MP)

This form of argument is called *Modus Ponens*, and abbreviated MP. We can apply 2.1.4 in our object language, **PL**, by constructing substitution instances of it. A substitution instance of a rule will be a set of wffs of **PL** that match, syntactically, the form of the rule. A substitution instance of MP will contain one wff whose main connective is a conditional and another which is precisely the antecedent of that conditional. The last wff of a substitution instance of MP will contain exactly the consequent of the conditional statement as a new wff in a derivation.

Notice that any substitution instance of MP will yield a valid argument. For, the only way to provide a counterexample to such an instance would be on a line on which the main operator of the conclusion were false and the main operator of the second premise were true. Any such valuation would make the first premise false and so make the inference valid. (Remember, a counterexample requires true premises and a false conclusion.)

`Given that every substitution instance of MP will be valid, we can substitute simple or complex formulas for α and β in 2.1.4 and be sure that the resulting deduction is valid. 2.1.5 is another example of MP.

2.1.5	$[(H \vee G) \supset I] \supset (K \bullet {\sim}L)$	
	$[(H \vee G) \supset I]$	/ $(K \bullet {\sim}L)$

Similar arguments will show that the forms 2.1.6 - 2.1.7 are also valid.

2.1.6	$\alpha \supset \beta$		
	${\sim}\beta$	/ ${\sim}\alpha$	Modus Tollens (MT)
2.1.7	$\alpha \vee \beta$		
	${\sim}\alpha$	/ β	Disjunctive Syllogism (DS)
2.1.8	$\alpha \supset \beta$		
	$\beta \supset \gamma$	/ $\alpha \supset \gamma$	Hypothetical Syllogism (HS)

You can check that each form is valid by using truth tables on the metalinguistic forms. For example, in 2.1.8, if we try to make the conclusion false, we have to make α true and γ false. Then, to make the first premise true, we have to make β true; that makes the second premise false. If we try to make the second premise true, by making β false, then we make the first premise false. In either case, we can not construct a counterexample. Thus, any substitution instance of HS will be valid.

For obvious reasons, we are mainly interested only in valid rules of inference. But it is sometimes useful to contrast the valid forms with invalid ones, like 2.1.9 and 2.1.10. Again, we can check them, using truth tables, or indirect truth tables.

2.1.9	$\alpha \supset \beta$		
	β	/ α	*Fallacy of Affirming the Consequent*
2.1.10	$\alpha \supset \beta$		*Fallacy of Denying the Antecedent*
	${\sim}\alpha$	/ ${\sim}\beta$	

To show that 2.1.9 is invalid, we can assign false to α and true to β. The premises turn out true and the conclusion turns out false! The same set of assignments provides a metalinguistic counterexample for 2.1.10. Any substitution instance of these forms will thus be invalid and we can construct an object-language counterexample in the same way.

Let's look at a couple of concrete instances to get an intuitive sense of the difference between valid and invalid arguments. Let 'P' stand for 'I study philosophy' and 'Q' stand for 'I write papers'. We can write the conditional 'P ⊃ Q' as 2.1.11.

2.1.11 If I study philosophy, then I write essays.

From 2.1.11 and the claim that I study philosophy, Modus Ponens licenses the inference that I write essays. From 2.1.11 and the claim that I do not write essays, Modus Tollens licenses the inference that I do not study philosophy. These are valid inferences. I would commit the fallacy of affirming the consequent if I concluded, from 2.1.11 and the claim that I write essays, that I study philosophy. Many people write papers without studying philosophy. Similarly, from 2.1.11 and the claim that I do not study philosophy, it does not follow that I do not write papers; such an inference would be an instance of the fallacy of denying the antecedent.

We have four rules of inference, now: MP, MT, DS, and HS. Let's see how to use them on the argument 2.1.12.

2.1.12	1. (X ⊃ Y) ⊃ T	
	2. S ∨ ~T	
	3. U ⊃ ~S	
	4. U	/ ~(X ⊃ Y)

We could show that the argument is valid using truth tables, including the indirect method. But, we can also show that it is valid by deriving the conclusion from the premises using our rules of inference. 2.1.13 is an example of the natural deductions we will use throughout the rest of the book.

2.1.13	1. (X ⊃ Y) ⊃ T		
	2. S ∨ ~T		
	3. U ⊃ ~S		
	4. U	/ ~(X ⊃ Y)	
	5. ~S	3, 4, MP	(taking 'U' for α and '~S' for β)
	6. ~T	2, 5, DS	(taking 'S' for α and '~T' for β)
	7. ~(X ⊃ Y)	1, 6, MT	(taking 'X ⊃ Y' for α and 'T' for β)
QED			

Here are some things to notice about 2.1.13. First, we number all of the premises as well as every wff that follows. While a derivation is really just the sequence of wffs, we will write our deductions in the metalanguage, including line numbers and justifications. The line numbers allow us to keep track of our justifications. All steps except the premises require justification. The justification of any step includes the line numbers and rule of inference used to generate the new wff. For example, '3, 4, MP' on line 5 indicates that '~S' is derived directly from the wffs at lines 3 and 4 by a use of the rule of Modus Ponens. The explanations such as "taking 'U' for α and '~S' for β are not required elements of the derivation, but they can be useful, especially when you are first learning to use natural deductions.

The conclusion of the argument is initially written after a single slash following the last premise. That conclusion, like the justifications of every following step, is not technically part of the deduction. Importantly, you may not use it as part of your proof. It merely indicates what the last numbered line of your derivation should be.

Lastly, 'QED' at the end of the derivation stands for 'Quod erat demonstratum', which is Latin for 'thus it has been shown'. 'QED' is a logician's punctuation mark: "I'm done!" It is not required, but looks neat and signals your intention to end the derivation. 2.1.14 is an example of a longer derivation using our first four rules of inference.

2.1.14

1. $\sim A \supset [A \vee (B \supset C)]$	
2. $(B \vee D) \supset \sim A$	
3. $B \vee D$	
4. $C \supset A$	/ D
5. $\sim A$	2, 3, MP
6. $A \vee (B \supset C)$	1, 5, MP
7. $B \supset C$	6, 5, DS
8. $B \supset A$	7, 4, HS
9. $\sim B$	8, 5, MT
10. D	3, 9, DS

QED

Constructing derivations can be intimidating at first. If you can, start with simple sentences, or negations of simple negations. Plan ahead. Feel free to work backwards from the conclusion to the premises on the side. For example, in 2.1.14, we could start the derivation by observing that we could get the conclusion, 'D', by DS from line 3 if we had '$\sim$B'. Then, both '$\sim$B' and 'D' are goals as we work forward through the proof.

Don't worry about introducing extraneous lines into your proof as long as they are the results of valid inferences. Especially as we introduce further rules, we are going to be able to infer statements which are not needed for the most concise derivation. But, as long as every step is valid, the entire inference will be valid. It is not the case that every wff must be used after it is introduced into the deduction.

Lastly, notice that some wffs may be used more than once in a derivation. In 2.1.14, the '$\sim$A' at line 5 was used first with premise 1 in a MP to yield the wff at line 6. Then, it is used immediately a second time, with the wff at line 6, to yield '$B \supset C$' on line 7.

Some students will have encountered proofs like these, perhaps in slightly less-rigorous form, in a geometry class, or in other mathematics courses. For other students, natural deductions of this sort are new. Be patient, and practice.

Exercises 2.1a. Derive the conclusions of each of the following arguments using natural deduction.

1.
1. $V \supset (W \supset X)$	
2. V	
3. $\sim X$	/ $\sim W$

2.
1. $X \supset Y$	
2. $\sim Y$	
3. $X \vee Z$	/ Z

3.
1. $E \supset F$	
2. $\sim F$	
3. $\sim E \supset (G \bullet H)$	/ $G \bullet H$

4.
1. $I \supset J$	
2. $J \supset K$	
3. $\sim K$	/ $\sim I$

5.
1. $T \supset S$	
2. $S \supset R$	
3. T	/ R

6.
1. $(I \bullet L) \supset (K \vee J)$	
2. $I \bullet L$	
3. $\sim K$	/ J

7.
1. $G \supset E$
2. $F \supset \sim E$
3. $H \vee F$
4. $\sim H$ / $\sim G$

8.
1. $\sim Q \supset (N \bullet O)$
2. $(N \bullet O) \supset (P \supset Q)$
3. $M \vee \sim Q$
4. $\sim M$ / $\sim P$

9.
1. $A \supset D$
2. $D \supset (B \supset C)$
3. B
4. A / C

10.
1. $L \vee N$
2. $\sim L$
3. $N \supset (M \vee O)$
4. $(M \vee O) \supset (P \equiv Q)$ / $P \equiv Q$

11.
1. $U \supset V$
2. $\sim V$
3. $U \vee W$
4. $W \supset X$ / X

12.
1. $X \supset Z$
2. $Z \supset Y$
3. $\sim Y$
4. $\sim X \supset A$ / A

13.
1. $C \supset B$
2. $B \supset D$
3. $(C \supset D) \supset E$ / E

14.
1. $E \supset H$
2. $G \vee \sim F$
3. $\sim G$
4. $H \supset F$ / $\sim E$

15.
1. $J \supset L$
2. $L \supset (I \bullet M)$
3. $(I \bullet M) \supset K$
4. $\sim K$ / $\sim J$

16.
1. $N \vee (Q \equiv R)$
2. $N \supset P$
3. $P \supset M$
4. $\sim M$ / $Q \equiv R$

17.
1. $N \vee (P \bullet \sim R)$
2. $(P \bullet \sim R) \supset Q$
3. $N \supset O$
4. $\sim O$ / Q

18.
1. $R \supset S$
2. $S \supset (T \vee U)$
3. R
4. $\sim T$ / U

19.
1. $Q \supset (\sim R \supset S)$
2. $T \vee Q$
3. $\sim T$
4. $R \supset T$ / S

20.
1. $C \supset (D \equiv \sim E)$
2. $(D \equiv \sim E) \supset (B \vee A)$
3. $C \supset \sim B$
4. C / A

21.
1. $\sim J \supset K$
2. $K \supset (L \supset M)$
3. $J \supset M$
4. $\sim M$ / $\sim L$

22.
1. $V \supset (W \vee U)$
2. $X \vee V$
3. $X \supset Y$
4. $\sim Y$
5. $\sim Y \supset \sim W$ / U

23.
1. $X \supset (Y \supset Z)$
2. $W \vee X$
3. $W \supset Y$
4. $\sim Y$
5. $\sim W \supset Y$ / Z

24.
1. $(H \bullet \sim G) \supset F$
2. $F \supset (G \vee J)$
3. $I \vee (H \bullet \sim G)$
4. $I \supset G$
5. $\sim G$ / J

25.
1. $A \supset B$
2. $B \supset (C \supset D)$
3. $E \vee C$
4. $E \supset B$
5. $\sim B$
6. $C \supset A$ / D

Exercises 2.1b. Translate each of the following paragraphs into arguments written in **PL**. Then, derive the conclusions of the arguments using the first four rules of our system of natural deduction.

1. If Allison doesn't go grocery shopping, Billy will go. Allison goes grocery shopping only if Carla gets home from school early. Carla doesn't get home early. Therefore, Billy goes grocery shopping.

2. Don Juan plays golf only if Edie makes a reservation. If Edie makes a reservation, then Frederique writes it on the calendar. Don Juan played golf. So Frederique wrote it down on the calendar.

3. If Gertrude mops the kitchen, then Hillary washes the dishes. Either Inez or Gertrude mops the kitchen. Inez doesn't mop the kitchen. So Hillary washes the dishes.

4. Katerina driving to practice is a necessary condition for Jelissa's playing soccer. Katerina drives only if Liza puts gas in her car. Liza doesn't put gas in the car. So, Jelissa doesn't play soccer.

5. Quinn or Raina will be valedictorian. Quinn's being valedictorian entails that she receives an A+ in Spanish. She doesn't receive an A+ in Spanish. So, Raina is valedictorian.

6. Nico skateboards if Mandy gives him lessons. If Nico skateboards, then either Olivia or Patricia will watch. Mandy gives skateboarding lessons. Olivia doesn't watch. So, Patricia watches.

7. Jose will play either trombone or ukulele. If he plays trombone, then he'll also play violin. If he plays ukulele, then he'll also play a woodwind instrument. He doesn't play violin. So, he plays a woodwind instrument.

8. Francine pays the bills only if Gerald balances the checkbook. Gerald balances the checkbook only if Esmeralda collects receipts. Either Hank spends money or Esmeralda doesn't collect receipts. Hank doesn't spend money, so Francine doesn't pay the bills.

9. If the corn doesn't grow, dandelions will grow. If dandelions grow, then the apple tree will bloom. If the corn grows, then the badgers will eat the crops. The badgers don't eat the crops. So, the apple tree blooms.

10. If the zoo has hippos, then it has yaks. If the zoo has yaks, then it has zebras. The zoo has either water buffalo or hippos. The zoo has water buffalo only if they have yaks. The zoo doesn't have yaks. So, the zoo has zebras.

§2.2: Rules of Inference 2

Here are four more valid forms which you can check using the indirect truth table method.

2.2.1	α		
	β	/ $\alpha \bullet \beta$	Conjunction (Conj)
2.2.2	α	/ $\alpha \vee \beta$	Addition (Add)
2.2.3	$\alpha \bullet \beta$	/ α	Simplification (Simp)
2.2.4	$(\alpha \supset \beta) \bullet (\gamma \supset \delta)$		
	$\alpha \vee \gamma$	/ $\beta \vee \delta$	Constructive Dilemma (CD)

Make sure to understand the difference between conjunction and addition. Conjunction allows us to put two prior premises together on one line. Addition allows us to add, using disjunction, any wff to one we have already established; if you have already established α, you have clearly established that either α or anything else holds. Be especially careful to avoid these two related but *invalid* inferences!

2.2.5	α	/ $\alpha \bullet \beta$
2.2.6	$\alpha \vee \beta$	/ α

Simplification only allows you to infer the first conjunct of a conjunction. A later rule of equivalence will allow us to infer the second conjunct. For now, our list of rules is incomplete. We must leave the second conjunct alone.

Lastly, note the similarity between Constructive Dilemma and Modus Ponens. Constructive Dilemma is a complex form of modus ponens: from the conjunction of two conditionals, and the disjunction of their antecedents, one can infer the disjunction of their consequents.

2.2.7 is a sample derivation using Conjunction and Simplification. 2.2.8 uses Addition.

2.2.7	1. $A \supset B$	
	2. $F \supset D$	
	3. $A \bullet E$	
	4. $\sim D$	/ $B \bullet \sim F$
	5. A	3, Simp
	6. B	1, 5, MP
	7. $\sim F$	2, 4, MT
	8. $B \bullet \sim F$	6, 7, Conj

QED

2.2.8	1. $\sim M \vee N$	
	2. $\sim\sim M$	/ $N \vee O$
	3. N	1, 2, DS
	4. $N \vee O$	3, Add

QED

2.2.9 is a simple derivation using CD. Note that one of the disjuncts used in the inference, at line 3, is itself a conjunction; the antecedent of the wff at line 2 is the same conjunction.

2.2.9	1. $N \supset (O \bullet P)$		
	2. $(Q \bullet R) \supset O$		
	3. $N \vee (Q \bullet R)$	$/ (O \bullet P) \vee O$	
	4. $[N \supset (O \bullet P)] \bullet [(Q \bullet R) \supset O]$		1, 2, Conj
	5. $(O \bullet P) \vee O$		4, 3, CD

QED

Lastly, 2.2.10 is just a slightly-longer derivation.

2.2.10	1. $(\sim A \vee B) \supset (G \supset D)$	
	2. $(G \vee E) \supset (\sim A \supset F)$	
	3. $A \vee G$	
	4. $\sim A$	$/ F \bullet D$
	5. G	3, 4, DS
	6. $G \vee E$	5, Add
	7. $\sim A \supset F$	2, 6, MP
	8. F	7, 4, MP
	9. $\sim A \vee B$	4, Add
	10. $G \supset D$	1, 9, MP
	11. D	10, 5, MP
	12. $F \bullet D$	8, 11, Conj

QED

Exercises 2.2a. For each of the following arguments, determine which, if any, of the eight rules of inference is being followed. If the inference is not in the form of one of the eight rules, it is invalid.

1. $A \supset (B \cdot C)$
   $\sim(B \cdot C)$ $\quad / \sim A$

2. $[(D \vee E) \supset F] \cdot [F \supset (G \equiv H)]$
   $(D \vee E) \vee F$ $\quad / F \vee (G \equiv H)$

3. $I \supset \sim J$
   $K \supset I$ $\quad / K \supset \sim J$

4. L
   $\sim M \cdot N$ $\quad / \sim(M \cdot N) \cdot L$

5. O $\quad / O \cdot \sim O$

6. P $\quad / P \vee [Q \equiv (R \cdot \sim P)]$

7. $S \vee \sim T$
   $\sim\sim T$ $\quad / \sim S$

8. $\sim U \equiv V$
   $(\sim U \equiv V) \supset W$ $\quad / W$

9. $X \supset \sim Y$
   $\sim Y \supset Z$ $\quad / (X \supset \sim Y) \cdot (\sim Y \supset Z)$

10. $(A \vee \sim B) \vee \sim\sim C$ $\quad / A \vee \sim B$

11. $\sim[D \supset (E \vee F)]$
    $[D \supset (E \vee F)] \vee [G \supset (E \cdot \sim F)]$
    $/ [G \supset (E \cdot \sim F)]$

12. $[(G \vee H) \cdot I] \cdot (\sim I \equiv K)$
    $/ (G \vee H) \cdot I$

Exercises 2.2b. Derive the conclusions of each of the following arguments using the eight rules of inference

1.
   1. A ⊃ (C • D)
   2. A • B / C

2.
   1. (M ⊃ N) • (O ⊃ P)
   2. M • Q / N ∨ P

3.
   1. I ∨ J
   2. ~I • K / J ∨ L

4.
   1. (F ∨ G) ⊃ H
   2. F • E / H

5.
   1. F ⊃ E
   2. ~E • G
   3. H / ~F • H

6.
   1. (~A ⊃ B) • (C ⊃ D)
   2. A ⊃ D
   3. ~D / B ∨ D

7.
   1. W ⊃ X
   2. ~X • Y / (~W ∨ Z) • ~X

8.
   1. T ∨ S
   2. ~T
   3. U / U • S

9.
   1. (V • W) ⊃ X
   2. V • Y
   3. W • Z / X

10.
    1. (E ∨ I) ⊃ H
    2. H ⊃ (F • G)
    3. E / (F • G) • E

11.
    1. (J ⊃ L) • (K ⊃ M)
    2. J • M
    3. ~L / M

12.
    1. N ∨ ~~P
    2. ~N • Q
    3. ~P ∨ Q / ~~P • Q

13.
    1. M ⊃ N
    2. N ⊃ O
    3. M • P / O ∨ P

14.
    1. W ⊃ Z
    2. Z ⊃ (X ∨ Y)
    3. W • Y
    4. (X ⊃ U) • (Y ⊃ V) / U ∨ V

15.
    1. B ⊃ A
    2. ~A • D
    3. ~B ⊃ C / C ∨ A

16.
    1. D ∨ E
    2. D ⊃ F
    3. ~F • G / (E ∨ H) • ~F

17.
    1. R ⊃ S
    2. S ⊃ (T ⊃ U)
    3. R
    4. U ⊃ R / T ⊃ R

18.
    1. (C ⊃ D) • (B ⊃ D)
    2. A • C
    3. A ⊃ C / D ∨ D

19.
    1. M ⊃ J
    2. (~M • ~J) ⊃ K
    3. ~J / K ∨ N

20.
    1. O ⊃ Q
    2. Q ⊃ P
    3. P ⊃ (R • S)
    4. O / R • S

21.
    1. (R ∨ T) ⊃ S
    2. S ⊃ U
    3. R / U ∨ T

22.
    1. I ⊃ J
    2. ~J • K
    3. ~J ⊃ L
    4. ~~I / K • L

23.
1. $Q \supset R$
2. $R \supset (S \vee T)$
3. Q
4. $\sim S \bullet U$ / $(T \bullet Q) \vee R$

24.
1. $(\sim V \supset W) \bullet (X \supset Y)$
2. $V \supset Z$
3. $\sim W \bullet X$
4. $\sim Z \bullet Y$ / $Y \bullet \sim V$

25.
1. $A \supset B$
2. $B \supset (C \supset D)$
3. $A \bullet D$
4. $\sim D$
5. $D \bullet E$ / E

Exercises 2.2c. Translate each of the following paragraphs into arguments written in **PL**. Then, derive the conclusions of the arguments using the eight rules of inference.

1. If Alessandro sings in the musical, then Beatriz will buy a ticket. Beatriz doesn't buy a ticket and Carlo goes to watch the musical. So, Alessandro doesn't sing in the musical and Beatriz doesn't buy a ticket.

2. If Don is an EMT then everyone is saved. All girls are saved provided that Frank is an EMT. Helga's being a doctor implies that Don is an EMT. Helga is a doctor; moreover all girls are saved. So, either everyone is saved or all girls are saved.

3. If Michelle eats ice cream, then she'll eat jelly beans. If she eats jelly beans, then she'll eat kettle corn. She eats ice cream; however she also eats lunch. So she'll eat either kettle corn or jelly beans.

4. If a classroom is quiet, then it is not rowdy. If a classroom isn't rowdy, then it's silent. The classroom is quiet and not rowdy. So, the classroom is quiet and silent.

5. Having a thunderstorm is a sufficient condition for needing an umbrella. Either it is very cloudy or you don't need an umbrella. It's not very cloudy. So, either there aren't thunderstorms or it's windy.

6. Either elephants or flamingos eat nuts. If elephants eat nuts, then gorillas eat fruit. Gorillas don't eat fruit but hippos eat berries. So, either flamingos eat nuts or hippos eat berries.

7. Mica goes swimming only if Nicole lifeguards. Pedro is free on the condition that Ona goes to the beach. Mica goes swimming unless Ona goes to the beach. Nicole doesn't lifeguard. So, either Pedro is free or Mica goes swimming.

8. Elia playing basketball is a necessary condition of her taking art. She'll walk the dog on the condition that she takes ceramics. She doesn't play basketball. She takes ceramics. So she doesn't take art but she does walk the dog.

9. Jaime either flies a kite or lies in the sun and listens to music. He doesn't fly a kite, but he juggles. If he lies in the sun, then he juggles. So, he either juggles or listens to music.

10. If Xavier takes Spanish, then Yolanda tutors him. Zeke pays Yolanda if she tutors Xavier. Either Waldo or Xavier take Spanish. Waldo doesn't take Spanish; also Yolanda doesn't tutor Xavier. So, Zeke pays Yolanda but Waldo doesn't take Spanish.

§2.3: Rules of Equivalence 1

Rules of Inference allow you to derive new conclusions based on previously accepted premises or derivations. They are justified by appeal to the truth-table definitions of validity. So, they must be used on whole lines only, and they go only one way: from premises to conclusion.

Rules of Equivalence, in contrast, allow you to substitute one proposition or part of a proposition with a logically equivalent expression. They are based on truth-table equivalences, and so may be used for any expressions, anywhere in a proof. They may be used on parts of lines, or on whole lines. They may be used in either direction. To check the legitimacy of the substitutions, we can use truth tables to show that the expressions are in fact logically equivalent. The appendix to this chapter provides complete truth tables proving most of the rules of equivalence.

De Morgan's Laws (DM)

$\sim(\alpha \bullet \beta) \rightleftharpoons \sim\alpha \lor \sim\beta$
$\sim(\alpha \lor \beta) \rightleftharpoons \sim\alpha \bullet \sim\beta$

Note the use of '$\rightleftharpoons$' to mean 'is logically equivalent to'. It is a symbol of the metalanguage, like the Greek letters, and not a symbol of the language of propositional logic. Also note that there are two versions of De Morgan's Law: one for the negation of a conjunction, and the other for the negation of a disjunction.

For all Rules of Equivalence, you can substitute any formula of the form of either side for a formula of the other; you can go forward (left-to-right) or backward (right-to-left). A forward DM distributes the tilde to the components of the conjunction or disjunction, changing the connective inside the parentheses. A backward DM factors out the tilde. Both the forward and the backward uses require the same justification. 2.3.1 contains a forward use of DM, while 2.3.2 contains a backward use.

2.3.1	1. $(A \lor B) \supset E$		
	2. $\sim E$		
	3. $A \lor D$	/ D	
	4. $\sim(A \lor B)$		1, 2, MT
	5. $\sim A \bullet \sim B$		4, DM
	6. $\sim A$		5, Simp
	7. D		3, 6, DS
QED			

2.3.2	1. $G \supset (H \bullet F)$		
	2. $\sim H \lor \sim F$	/ $\sim G$	
	3. $\sim(H \bullet F)$		2, DM
	4. $\sim G$		1, 3, MT
QED			

Association (Assoc)

$\alpha \vee (\beta \vee \gamma) \rightleftarrows (\alpha \vee \beta) \vee \gamma$
$\alpha \bullet (\beta \bullet \gamma) \rightleftarrows (\alpha \bullet \beta) \bullet \gamma$

As with DM, Assoc has a version for conjunction and a version for disjunction. Unlike DM, Assoc requires no switching of connectives. It merely allows you to re-group the component propositions and the two connectives must be the same. Remember, like all rules of equivalence, it can be used in either direction. Assoc is often used with DS, as in 2.3.3.

2.3.3	1. $(L \vee M) \vee N$	
	2. $\sim L$	
	3. $(M \vee N) \supset O$	/ O
	4. $L \vee (M \vee N)$	1, Assoc
	5. $M \vee N$	4, 2, DS
	6. O	3, 5, MP

QED

Distribution (Dist)

$\alpha \bullet (\beta \vee \gamma) \rightleftarrows (\alpha \bullet \beta) \vee (\alpha \bullet \gamma)$
$\alpha \vee (\beta \bullet \gamma) \rightleftarrows (\alpha \vee \beta) \bullet (\alpha \vee \gamma)$

The rule of distribution allows you to distribute a conjunction over a disjunction or to distribute a disjunction over a conjunction. The main operator is always switched (between conjunction and disjunction) after a use of Dist. So, using Dist on a sentence whose main operator is a disjunction yields a conjunction from which you can simplify!

Notice that while the grouping of terms changes, the order of the first two connectives remains after using a dist, with an extra connective of the first type added at the end (going left to right) or taken away (going right to left). So $\bullet\vee$ becomes $\bullet\vee\bullet$ and $\vee\bullet$ becomes $\vee\bullet\vee$ (or vice versa).

Be careful to distinguish Dist from Assoc. Assoc is used when you have two of the same connectives. Dist is used when you have a combination of conjunction and disjunction. 2.3.4 contains a forward use of Dist, while 2.3.5 contains a backward use.

2.3.4	1. $H \bullet (I \vee J)$	
	2. $\sim(H \bullet I)$ / $H \bullet J$	
	3. $(H \bullet I) \vee (H \bullet J)$	1, Dist
	4. $H \bullet J$	3, 2, DS

QED

2.3.5	1. $(P \vee Q) \bullet (P \vee R)$	
	2. $\sim P$ / $Q \bullet R$	
	3. $P \vee (Q \bullet R)$	1, Dist
	4. $Q \bullet R$	3, 2, DS

QED

Commutativity (Com)

$\alpha \vee \beta \rightleftarrows \beta \vee \alpha$
$\alpha \bullet \beta \rightleftarrows \beta \bullet \alpha$

Commutativity finally allows us to make some inferences that it is easy to see are valid. In effect, it doubles the rules DS, Simp, and Add. From a disjunction, we can now infer the first disjunct from the negation of the second. From a conjunction, we can now infer the second conjunct using Simp. And, we can add a proposition in front of a given wff. 2.3.6, 2.3.7, and 2.3.8 provide brief examples of each kind of inference.

2.3.6	1. P ∨ Q	
	2. ~Q	
	3. Q ∨ P	1, Com
	4. P	3, 4, DS

2.3.7	1. P • Q	
	2. Q • P	1, Com
	3. Q	2, Simp

2.3.8	1. P	
	2. P ∨ Q	1, Add
	3. Q ∨ P	2, Com

Each of the three derivations 2.3.6 - 2.3.8 can be inserted into any derivation. 2.3.9 demonstrates the use of commutativity with simplification and disjunctive syllogism.

2.3.9	1. A • B	
	2. B ⊃ (D ∨ E)	
	3. ~E / D	
	4. B • A	1, Com
	5. B	4, Simp
	6. D ∨ E	2, 5, MP
	7. E ∨ D	6, Com
	8. D	7, 3, DS

QED

Double Negation (DN)

$\alpha \rightleftarrows {\sim}{\sim}\alpha$

Double negation illustrates the difference between the negation of a statement and a statement with the opposite truth value of a given statement. The negation of a statement is precisely that statement with one additional tilde. '~P' is the negation of 'P'. But, 'P' is not the negation of '~P', even though it has the opposite truth value. The negation of '~P' is '~~P'. Later, when we get to indirect proof, the difference between negations and statements with opposite truth values will be important.

Double negation is often used right-to-left as a way of clearing extraneous tildes. But be careful not to add or subtract single tildes. They must be added or removed in consecutive pairs.

There are three ways to use DN to add two tildes to a statement with a binary connective. 2.3.10 can be transformed, in a single use of DN, into 2.3.11, 2.3.12, or 2.3.13.

2.3.10	P ∨ Q	
2.3.11	~~P ∨ Q	by double-negating the ‘P’
2.3.12	P ∨ ~~Q	by double-negating the ‘Q’
2.3.13	~~(P ∨ Q)	by double-negating the whole disjunction

DN, like Com, allows us to expand our uses of other rules, as we can see in 2.3.14.

2.3.14	1. ~F ⊃ ~G	
	2. G	
	3. F ⊃ H	/ H
	4. ~~G	2, DN
	5. ~~F	1, 4, MT
	6. F	5, DN
	5. H	3, 6, MP
QED		

Be careful to distinguish the rules of equivalence, which we saw in this section, from the rules of inference, which we saw in the previous two sections. One difference is that each rule of equivalence can be used in two different directions. Another difference is that the rules of equivalence are justified by showing that expressions of each form are logically equivalent, as I do for most of the rules of equivalence in the appendix to this chapter. A third difference is that rules of equivalence apply to any part of a proof, not just to whole lines. Rules of inference must be used on whole lines. Thus, 2.3.15 is an unacceptable use of MP, though we will be able to infer the desired wff in several steps, later.

2.3.15	P ⊃ (Q ⊃ R)
	Q
	P ⊃ R

In contrast, we can use any rule of equivalence on only a part of a line, as with DM in 2.3.16 and DN and DM in 2.3.17.

2.3.16	P ⊃ ~(Q ∨ P)	
	P ⊃ (~Q • ~P)	DM

2.3.17	S ⊃ (~P • Q)	
	S ⊃ (~P • ~~Q)	DN
	S ⊃ ~(P ∨ ~Q)	DM

Exercises 2.3a. Derive the conclusions of each of the following arguments using the rules of inference and the first five rules of equivalence.

1.
1. A ⊃ B
2. C • A / B

2.
1. (A • B) ∨ (A • C)
2. D ⊃ ~A / ~D

3.
1. E ⊃ F
2. ~ ~E • G / F

4.
1. H ∨ J
2. I • ~H / J

5.
1. X ⊃ Y
2. Z • Y / ~X

6.
1. F ⊃ (C ∨ D)
2. ~[C ∨ (D ∨ E)] / ~F

7.
1. X ⊃ Y
2. (~Y • Z) • T
3. X ∨ W / W

8.
1. ~A ∨ B
2. ~[(~A ∨ C) ∨ D] / B

9.
1. R • (S ∨ T)
2. ~R ∨ ~S / T

10.
1. I • {~[J • (K ∨ L)] • M}
2. (~J ∨ ~L) ⊃ N / N

11.
1. J ⊃ K
2. K ⊃ [L ∨ (M • N)]
3. ~N • J / L

12.
1. ~[(G • H) • I]
2. G • I / ~H

13.
1. Q ⊃ R
2. ~(S ∨ T)
3. T ∨ Q / R

14.
1. A ∨ (B • C)
2. (C ∨ A) ⊃ ~ ~B / B

15.
1. (K • L) • M
2. K ⊃ N
3. N ⊃ ~(O ∨ P) / ~P

16.
1. [O ∨ (P • Q)] ⊃ R
2. R ⊃ ~S
3. P • S / ~Q

17.
1. E ⊃ F
2. F ⊃ ~(G ∨ H)
3. I • E / ~H

18.
1. T ∨ (U • V)
2. T ⊃ (W • X)
3. ~W / V

19.
1. A ⊃ B
2. ~[(C • D) ∨ (C • B)]
3. C • E / ~A

20.
1. [T • (U ∨ V)] ⊃ W
2. W ⊃ ~X
3. Y • X / ~(T • U) • ~(T • V)

21.
1. F ⊃ G
2. H ⊃ I
3. (J ∨ F) ∨ H
4. ~J • ~G / I

22.
1. O ⊃ P
2. (O • ~Q) • ~R
3. P ⊃ [Q ∨ (R ∨ S] / S

23.
1. U ⊃ V
2. V ⊃ ~(W • X)
3. U • (W • Y) / ~X • Y

24.
1. K ⊃ ~L
2. K ∨ (M • N)
3. M ⊃ ~N / ~(L • N)

25.
1. (O • P) ⊃ (Q • R)
2. (P ⊃ ~Q) • (O ⊃ ~R)
3. P / ~P ∨ ~ O

Exercises 2.3b. Translate each of the following paragraphs into arguments written in **PL**. Then, derive the conclusions of the arguments using the eight rules of inference and the first five rules of equivalence.

1. If Albert asks Bernice on a date, then she'll say yes. Bernice doesn't say yes to a date and her cat died, but her dog is still alive. So, Albert didn't ask Bernice on a date.

2. Callie majors in English only if she reads Charles Dickens. Either Callie and Elisa major in English or Callie and Franz major in English. So, Callie reads Charles Dickens.

3. If there is a mouse in the house, then nuts were left out. The lights were turned off unless no nuts were left out. Neither the lights were turned off nor were the doors left open. So, there was no mouse in the house.

4. It is not the case that either there was a paper or both a quiz and recitation in French class. If there is no quiz, then the students are happy, and if there is no recitation, the teacher is happy. So, either the students or the teacher is happy.

5. Roland will either go on the upside-down roller coaster, or the speedy vehicle or the water slide. He doesn't go on the upside-down roller coaster and he doesn't go on the speedy vehicle. If he goes on the tilt-a-whirl, then he won't go on the water slide. So, he doesn't go on the tilt-a-whirl.

6. Mario either gets an x-ray, or Yasmin and Zoe take care of him. If Mario gets an x-ray, then Zoe will take care of him. It is not the case that Winnie or Zoe take care of Mario. So Yasmin and Zoe take care of Mario.

7. After-school activities running late entails that the buses will run late. The buses running late is a sufficient condition for either Carlos and Deandra getting home late or Edna missing dinner. Either after-school activities run late but Carlos doesn't get home late, or after-school activities run late but Deandra doesn't get home late. So, Edna misses dinner.

8. If Luz doesn't travel to Greece, the she'll go to Haiti. She'll go to Israel given that she travels to Haiti. She doesn't go to either Greece or Jordan. So she goes to Israel and not Jordan.

9. It is not the case that either Ernesto and Francisco go to swim practice or Gillian or Hayden go to swim practice. Either Isaac or Joan goes to swim practice. If Isaac goes to swim practice, then Hayden will go to swim practice. So, Joan goes to swim practice.

10. If it's not the case that both Katrina and Laetitia go to math class, then Ms. Macdonald will be angry. Ms. Macdonald is angry only when Nigel skips math class. It is not the case that either Olivia and Polly both skip math class, or Nigel does. Therefore, Laetitia goes to math class.

§2.4: Rules of Equivalence 2

This section introduces the final five of our eighteen rules of equivalence.

Contraposition (Cont)

$\alpha \supset \beta \rightleftarrows \sim\beta \supset \sim\alpha$

Cont allows you to switch a conditional for its contrapositive. In other words, the antecedent and consequent of a conditional statement may be exchanged if they are both negated (or, right-to-left, un-negated). Cont is often used with HS, as in 2.4.1.

2.4.1	1. $A \supset B$		
	2. $D \supset \sim B$	/ $A \supset \sim D$	
	3. $\sim\sim B \supset \sim D$		2, Cont
	4. $B \supset \sim D$		3, DN
	5. $A \supset \sim D$		1, 4, HS
QED			

Cont can be tricky when only one formula is negated, as we can see in 2.4.2 and 2.4.3, which perform the same transformation in different orders. You can either add a negation to both the antecedent and consequent when you use Cont, or you can take a tilde off of each of them. But, you can not mix-and-match. Thus, you often need to invoke DN together with Cont.

2.4.2	$A \supset \sim B$	
	$\sim\sim B \supset \sim A$	by Cont (left-to-right)
	$B \supset \sim A$	by DN

2.4.3	$A \supset \sim B$	
	$\sim\sim A \supset \sim B$	by DN
	$B \supset \sim A$	by Cont (right-to-left)

Material Implication (Impl)

$\alpha \supset \beta \rightleftarrows \sim\alpha \vee \beta$

The rule of material implication allows you to change a disjunction to a conditional or vice versa. It is often easier to work with disjunctions. From a disjunction, you may be able to use De Morgan's laws to get a conjunction. You may be able to use distribution, which does not apply to conditionals. In contrast, sometimes you just want to work with conditionals, using hypothetical syllogism, modus ponens, or modus tollens. Proofs are over-determined by our system: there are multiple ways to do them once we have all the rules. The rule of material implication gives us a lot of options.

The rule of material implication also illustrates the underlying logic of the material conditional. It is just a way of saying that either the antecedent is false or the negation is true. Unlike many natural-language conditionals, it says nothing about the connections between the antecedent and the consequent.

The derivation 2.4.4 illustrates the use of Impl with HS.

2.4.4	1. $G \supset \sim E$	
	2. $E \vee F$	/ $G \supset F$
	3. $\sim\sim E \vee F$	2, DN
	4. $\sim E \supset F$	3, Impl
	5. $G \supset F$	1, 4, HS
	QED	

Material Equivalence (Equiv)

$\alpha \equiv \beta \rightleftarrows (\alpha \supset \beta) \bullet (\beta \supset \alpha)$
$\alpha \equiv \beta \rightleftarrows (\alpha \bullet \beta) \vee (\sim\alpha \bullet \sim\beta)$

Equiv is almost the only thing you can do with a biconditional. There are two distinct versions. If you have a biconditional in your premises, you can unpack it in either way. The first version tends to be more useful since it yields a conjunction both sides of which you can simplify, as in 2.4.5

2.4.5	1. $A \equiv B$	
	2. $\sim A$	
	3. $B \supset C$ / $\sim B \bullet (A \supset C)$	
	4. $(A \supset B) \bullet (B \supset A)$	1, Equiv
	5. $(B \supset A) \bullet (A \supset B)$	4, Com
	6. $B \supset A$	5, Simp
	7. $\sim B$	6, 2, MT
	8. $A \supset B$	4, Simp
	9. $A \supset C$	8, 3, HS
	10. $\sim B \bullet (A \supset C)$	7, 9, Conj

The second version of material equivalence reflects the truth-table definition of the connective. Remember, a biconditional is true if either both components are true (first disjunct) or both disjuncts are false (second disjunct). 2.4.6 demonstrates an instance of the second use of the rule.

2.4.6	1. $D \equiv E$	
	2. $\sim D$ / $\sim D \bullet \sim E$	
	3. $(D \bullet E) \vee (\sim D \bullet \sim E)$	1, Equiv
	4. $\sim D \vee \sim E$	2, Add
	5. $\sim(D \bullet E)$	4, DM
	6. $\sim D \bullet \sim E$	3, 5, DS

If you need to derive a biconditional, again the first version of the rule is often more useful. First, derive the two component conditionals. Then conjoin them and use the rule. Take a moment to make sure you see how the rule is used at 2.4.7.

2.4.7	1. $\sim[(K \supset \sim H) \bullet (\sim H \supset K)]$	
	2. $(I \bullet J) \supset (K \equiv \sim H)$	/ $\sim(I \bullet J)$
	3. $\sim(K \equiv \sim H)$	1, Equiv
	4. $\sim(I \bullet J)$	2, 3, MT
	QED	

Exportation (Exp)

$\alpha \supset (\beta \supset \gamma) \rightleftarrows (\alpha \bullet \beta) \supset \gamma$

Exportation allows you to group antecedents of nested conditionals either together as a conjunction (on the right) or separately (on the left). According to exportation, a typical nested conditional like 2.4.8 can be translated as either 2.4.9 or 2.4.10.

2.4.8	If I get my paycheck today, then if you come with me, we can go to dinner.
2.4.9	$P \supset (C \supset D)$
2.4.10	$(P \bullet C) \supset D$

While 2.4.9 is the more natural reading of 2.4.8, the alternative 2.4.10 is also satisfying. A close English translation of 2.4.10, at 2.4.11, is intuitively equivalent to the original.

2.4.11 If I get my paycheck today and you come with me, then we can go to dinner.

Further, exportation, when combined with commutativity, allows us to switch antecedents. So 2.4.9 is also equivalent to 2.4.12. A natural translation of that proposition into English is at 2.4.13.

2.4.12	$C \supset (P \supset D)$
2.4.13	If you come with me, then if I get my paycheck, we can go to dinner.

While 2.4.13 is not as intuitively satisfying as 2.4.11 as an equivalent of 2.4.8, they are all logically equivalent. The difference in tone or presupposition arises from the awkwardness of representing natural-language conditionals, and their causal properties, with the material conditional.

The rule of exportation sometimes allows you get to MP or MT, as in 2.4.14.

2.4.14	1. $L \supset (M \supset N)$	
	2. $\sim N$	/ $\sim L \vee \sim M$
	3. $(L \bullet M) \supset N$	1, Exp
	4. $\sim(L \bullet M)$	3, 2, MT
	5. $\sim L \vee \sim M$	4, DM

QED

Tautology (Taut)

$\alpha \rightleftarrows \alpha \bullet \alpha$

$\alpha \rightleftarrows \alpha \vee \alpha$

Tautology eliminates redundancy. The conjunction version is redundant right-to-left, since we can use Simp instead. The disjunction version is redundant left-to-right, since we can use Add instead. But, the other directions can be useful, especially for disjunction, as in 2.4.15.

2.4.15	1. $O \supset \sim O$	/ $\sim O$
	2. $\sim O \vee \sim O$	1, Impl
	3. $\sim O$	2, Taut

QED

Some more potentially helpful examples

We have now seen all eighteen of our rules. Some of the proofs you will be asked to derive now are long. Some are quite difficult. In the next few sections, I will introduce two additional proof techniques which will make derivations easier.

For now, you can make many derivations simpler by learning some simple techniques which can be applied in various different proofs. 2.4.16 - 2.4.22 contain some derivations and useful tricks that may be adapted to other, longer proofs.

2.4.16	1. ~A	/ A ⊃ B
	2. ~A ∨ B	1, Add
	3. A ⊃ B	2, Impl
QED		

2.4.17	1. E	/ F ⊃ E
	2. E ∨ ~F	1, Add
	3. ~F ∨ E	2, Com
	3. F ⊃ E	3, Impl
QED		

2.4.18	1. G ⊃ (H ⊃ I)	/ H ⊃ (G ⊃ I)
	2. (G • H) ⊃ I	1, Exp
	3. (H • G) ⊃ I	2, Com
	4. H ⊃ (G ⊃ I)	3, Exp
QED		

2.4.19	1. O ⊃ (P • Q)	/ O ⊃ P
	2. ~O ∨ (P • Q)	1, Impl
	3. (~O ∨ P) • (~O ∨ Q)	2, Dist
	4. ~O ∨ P	3, Simp
	5. O ⊃ P	4, Impl
QED		

2.4.20	1. (R ∨ S) ⊃ T	/ R ⊃ T
	2. ~(R ∨ S) ∨ T	1, Impl
	3. (~R • ~S) ∨ T	2, DM
	4. T ∨ (~R • ~S)	3, Com
	5. (T ∨ ~R) • (T ∨ ~S)	4, Dist
	6. T ∨ ~R	5, Simp
	7. ~R ∨ T	6, Com
	8. R ⊃ T	7, Impl
QED		

2.4.21

1. $W \supset X$		
2. $Y \supset X$	/ $(W \vee Y) \supset X$	
3. $(W \supset X) \bullet (Y \supset X)$		1, 2, Conj
4. $(\sim W \vee X) \bullet (Y \supset X)$		3, Impl
5. $(\sim W \vee X) \bullet (\sim Y \vee X)$		4, Impl
6. $(X \vee \sim W) \bullet (\sim Y \vee X)$		5, Com
7. $(X \vee \sim W) \bullet (X \vee \sim Y)$		6, Com
8. $X \vee (\sim W \bullet \sim Y)$		7, Dist
9. $(\sim W \bullet \sim Y) \vee X$		8, Com
10. $\sim(W \vee Y) \vee X$		9, DM
11. $(W \vee Y) \supset X$		10, Impl

QED

2.4.22

1. $(J \vee K) \supset (L \bullet M)$	
2. $\sim J \supset (N \supset \sim N)$	
3. $\sim L$	/ $\sim N$
4. $\sim L \vee \sim M$	3, Add
5. $\sim(L \bullet M)$	4, DM
6. $\sim(J \vee K)$	1, 5, MT
7. $\sim J \bullet \sim K$	6, DM
8. $\sim J$	7, Simp
9. $N \supset \sim N$	2, 8, MP
10. $\sim N \vee \sim N$	9, Impl
11. $\sim N$	10, Taut

QED

Exercises 2.4a. Derive the conclusions of each of the following arguments using the rules of inference and equivalence.

1.
1. $A \supset B$
2. $B \supset \sim B$ / $\sim A$

2.
1. $\sim K \vee L$
2. $L \supset \sim K$ /$\sim K$

3.
1. $(A \supset B) \supset C$
2. $\sim A \vee (B \bullet D)$ / C

4.
1. $G \supset H$
2. $\sim(I \supset H)$ / $\sim G$

5.
1. $\sim I \vee J$
2. $J \equiv K$
3. $(I \bullet L) \vee (I \bullet M)$ / K

6.
1. $(T \bullet U) \supset V$
2. $\sim(T \supset W)$ / $U \supset V$

7.
1. $W \supset (X \bullet Y)$
2. $(W \bullet \sim X) \vee Z$ / Z

8.
1. $N \supset (O \bullet P)$
2. $\sim N \supset Q$ / $\sim O \supset Q$

9.
1. $E \equiv F$
2. $\sim(G \vee E)$ / $\sim F$

10.
1. $G \vee H$
2. $\sim I \bullet (J \bullet \sim G)$ / $H \vee \sim I$

11.
1. $A \vee (B \vee A)$
2. $\sim(B \vee C)$
3. $A \supset D$ / D

12.
1. $(H \bullet I) \supset J$
2. $H \bullet (I \vee K)$ / $\sim J \supset K$

13.
1. $L \supset {\sim}({\sim}M \vee K)$
2. $M \supset ({\sim}K \supset N)$
3. ${\sim}N$ / ${\sim}L$

14.
1. $Q \supset R$
2. $R \supset (S \supset T)$ / ${\sim}T \supset (S \supset {\sim}Q)$

15.
1. $D \equiv E$
2. $(E \vee F) \supset G$
3. ${\sim}(G \vee H)$ / ${\sim}D$

16.
1. $D \vee (E \vee F)$
2. $F \supset (G \bullet H)$
3. ${\sim}G$ / $D \vee E$

17.
1. $(X \supset Y) \supset Z$
2. $W \supset {\sim}Z$ / ${\sim}(W \bullet Y)$

18.
1. $R \supset T$
2. $T \supset S$
3. $(U \bullet S) \supset R$
4. ${\sim}({\sim}U \vee T)$ / $R \equiv S$

19.
1. $(S \equiv T) \bullet {\sim}U$
2. ${\sim}S \vee ({\sim}T \vee U)$ / ${\sim}S$

20.
1. $[V \vee (W \vee X)] \supset Y$
2. $Y \supset Z$ / $Z \vee {\sim}V$

21.
1. $A \supset (B \supset C)$
2. ${\sim}C \vee (D \bullet E)$
3. ${\sim}(D \vee F)$ / ${\sim}A \vee {\sim}B$

22.
1. $F \supset (G \supset H)$
2. $G \bullet {\sim}H$
3. $J \supset F$ / ${\sim}J$

23.
1. $N \supset O$
2. $P \supset Q$
3. ${\sim}(Q \vee O)$ / $P \equiv N$

24.
1. $O \equiv P$
2. $P \equiv Q$ / $O \equiv Q$

25.
1. $T \supset (U \supset V)$
2. $Q \supset (R \supset V)$
3. $(T \bullet U) \vee (Q \bullet R)$ / V

26.
1. ${\sim}(X \supset Y)$
2. $Y \vee (Z \bullet A)$ / $Z \equiv A$

Exercises 2.4b. Translate each of the following paragraphs into arguments written in **PL**. Then, derive the conclusions of the arguments using the rules of inference and equivalence.

1. There is a rainbow if, and only if, the sun is out. The sun is not out. So, there is no rainbow.

2. If there are alpacas on the farm, then there are beagles. If there are beagles, then there are cows. So, either there are cows or there are no alpacas.

3. If David quits the team, then Sandra watches the games provided that Ross joins the team. So, it is not the case that David quits the team, and Ross joins the team, and Sandra doesn't watch the games.

4. If there is a line, Marla must wait in it. In New England High School shows up, then there is a line if the organist attends. The organist attends and New England High School shows up. Therefore, Marla must wait in line.

5. Cecilia goes roller skating if and only if Denise comes with her. Denise and Elise go roller skating, and Felicia goes running. So, Cecilia goes roller skating.

6. If you are from the planet Orc, then you have pin-sized nostrils. But, things with pin-sized nostrils are not from Orc. Either you are from Orc or Quaznic, or you rode a long way on your spaceship. So, you are from Quaznic unless you rode a long way on your spaceship.

7. It is not the case that violets bloom only if they are watered. Either violets are watered or they undergo special treatment. So, they undergo special treatment.

8. The Janitor cleans the school if and only if Kurt pays him. So Kurt doesn't pay him if, and only if, the janitor doesn't clean the school.

9. Either Ana doesn't like lemons, or she likes mangoes. She likes lemons and nectarines, and oranges. She either doesn't like mangoes, or she likes plums. So, she likes plums.

10. If Francesca playing the xylophone entails that she yawns in class then Zara gives a presentation in class. If Zara gives a presentation, then the woodwind players listen. So either the woodwind players listen or Francesca plays xylophone.

§2.5: Conditional Proof

There are three more sections in this chapter on natural deductions in **PL**. This section introduces a proof method, called conditional proof, which allows us to simplify many long, difficult proofs. It will also allow us to derive logical truths, or theorems of our system of logic, as we will see in the next section. In the last section of the chapter, we will examine a third method of proof, indirect proof.

When you want to derive a conditional conclusion, you can assume the antecedent of the conditional, for the purposes of the derivation, taking care to indicate the presence of that assumption later. Consider the argument at 2.5.1, which has a conditional conclusion.

2.5.1	1. A ∨ B	
	2. B ⊃ (E • D)	/ ~A ⊃ D

Think about what would happen if we had the antecedent of the conditional conclusion, '~A', as another premise. First, we would be able to infer 'B' by DS with line 1. Then, since we would have 'B', we could use MP to infer 'E • D' from line 2. Lastly, given 'E • D' we could use Com and Simp and get 'D'. So, 'D' would follow from '~A'. The rule of conditional proof formalizes this line of thought.

Method for Conditional Proof

1. Indent, assuming the antecedent of your desired conditional.
   Write 'ACP', for 'assumption for conditional proof'.
   Use a vertical line to set off the assumption from the rest of your derivation.
2. Derive the consequent of desired conditional.
   Continue the vertical line.
   Proceed as you would normally, using any lines already established.
3. Discharge (un-indent).
   Write the first line of your assumption, a ⊃, and the last line of the indented sequence.
   Justify the un-indented line with CP, and indicate the indented line numbers.

The line of thought we took discussing 2.5.1 is thus formalized by using the indented sequence you see at 2.5.2.

2.5.2	1. A ∨ B		
	2. B ⊃ (E • D)	/ ~A ⊃ D	
		3. ~A	ACP	Suppose ~A.
		4. B	1, 3, DS	
		5. E • D	2, 4, MP	
		6. D • E	5, Com	
		7. D	6, Simp	Then D would follow.
	8. ~A ⊃ D	3-7, CP	So, if ~A were true, then D would be.
	QED		

The purpose of indenting and using a vertical line is to mark the scope of your assumption. Any statements you derive within the scope of that assumption are not proven, but only derived from that assumption. Thus, after you discharge your assumption, you *may not* use statements within the scope of that assumption later in the proof. You could have discharged your assumption after any number of steps in the indented sequence: '~A ⊃ (D • E)'; '~A ⊃ (E • D)'; '~A ⊃ B'; and even '~A ⊃ ~A' are all valid inferences given the premises. But none of the consequents of those conditional statements are validly inferred from the premises.

Conditional proof makes many of the derivations we have done significantly easier. Let's call the method of proof we have been using all along the direct method. Contrast the same argument proved directly, in 2.5.3, and conditionally, in 2.5.4.

2.5.3	1. (P ⊃ Q) • (R ⊃ S)	/ (P • R) ⊃ (Q • S)	Direct Method
	2. P ⊃ Q	1, Simp	
	3. ~P ∨ Q	2, Impl	
	4. (~P ∨ Q) ∨ ~R	3, Add	
	5. ~P ∨ (Q ∨ ~R)	4, Assoc	
	6. (R ⊃ S) • (P ⊃ Q)	1, Com	
	7. (R ⊃ S)	6, Simp	
	8. ~R ∨ S	7, Impl	
	9. (~R ∨ S) ∨ ~P	8, Add	
	10. ~P ∨ (~R ∨ S)	9, Com	
	11. [~P ∨ (Q ∨ ~R)] • [~P ∨ (~R ∨ S)]	5, 10, Conj	
	12. ~P ∨ [(Q ∨ ~R) • (~R ∨ S)]	11, Dist	
	13. ~P ∨ [(~R ∨ Q) • (~R ∨ S)]	12, Com	
	14. ~P ∨ [~R ∨ (Q • S)]	13, Dist	
	15. P ⊃ [~R ∨ (Q • S)]	14, Impl	
	16. P ⊃ [R ⊃ (Q • S)]	15, Impl	
	17. (P • R) ⊃ (Q • S)	16, Exp	

QED

2.5.4	1. (P ⊃ Q) • (R ⊃ S)	/ (P • R) ⊃ (Q • S)	Conditional Method
	│2. P • R	ACP	
	│3. P ⊃ Q	1, Simp	
	│4. P	2, Simp	
	│5. Q	3, 4, MP	
	│6. (R ⊃ S) • (P ⊃ Q)	1, Com	
	│7. R ⊃ S	6, Simp	
	│8. R • P	2, Com	
	│9. R	8, Simp	
	│10. S	7,9, MP	
	│11. Q • S	5, 10, Conj	
	12. (P • R) ⊃ (Q • S)	2-11, CP	

QED

Not only is the conditional method of proof ordinarily shorter, as in this case. It is also conceptually much less difficult. In this case, to see that one has to add what one needs in the direct version is not easy. The conditional proof proceeds in obvious ways.

You can use CP repeatedly within the same proof, whether nested or sequentially. 2.5.5 demonstrates a nested use of CP while 2.5.6 shows how we can use CP sequentially to prove biconditionals.

2.5.5	1. $P \supset (Q \vee R)$		
	2. $(S \bullet P) \supset {\sim}Q$	/ $(S \supset P) \supset (S \supset R)$	
	$\vert$ 3. $S \supset P$	ACP	Now we want $S \supset R$.
	$\vert$ $\vert$ 4. S	ACP	Now we want R.
	$\vert$ $\vert$ 5. P	3, 4, MP	
	$\vert$ $\vert$ 6. $Q \vee R$	1, 5, MP	
	$\vert$ $\vert$ 7. $S \bullet P$	4, 5, Conj	
	$\vert$ $\vert$ 8. ${\sim}Q$	2, 7, MP	
	$\vert$ $\vert$ 9. R	6, 8, DS	
	$\vert$ 10. $S \supset R$	4-9, CP	
	11. $(S \supset P) \supset (S \supset R)$	3-10, CP	

QED

2.5.6	1. $(B \vee A) \supset D$	
	2. $A \supset {\sim}D$	
	3. ${\sim}A \supset B$	/ $B \equiv D$
	$\vert$ 4. B	ACP
	$\vert$ 5. $B \vee A$	4, Add
	$\vert$ 6. D	1, 5, MP
	7. $B \supset D$	4-6 CP
	$\vert$ 8. D	ACP
	$\vert$ 9. ${\sim}{\sim}D$	8, DN
	$\vert$ 10. ${\sim}A$	2, 9, MT
	$\vert$ 11. B	3, 10, MP
	12. $D \supset B$	8-11 CP
	13. $(B \supset D) \bullet (D \supset B)$	7, 12, Conj
	14. $B \equiv D$	13, Equiv

QED

Notice that we start the second sequence at 2.5.6 intending to derive ‘B’. We already have a ‘B’ in the proof, though, at line 4. But that ‘B’ was a discharged assumption, and is off-limits after line 6.

2.5.7 demonstrates a standard method for proving biconditionals. In such cases, you want ‘$P \equiv Q$’ which is logically equivalent to ‘$(P \supset Q) \bullet (Q \supset P)$’. This method is not always the best one, but it is usually a good first thought.

Method for Proving A Biconditional Conclusion

Assume P, Derive Q, Discharge.
Assume Q, Derive P, Discharge.
Conjoin the two conditionals.
Use Material Equivalence to yield the biconditional.

You may also use CP in the middle of a proof to derive statements which are not your main conclusion, as in 2.5.7.

2.5.7	1. P ⊃ (Q • R)	
	2. (P ⊃ R) ⊃ (S • T)	/ T
	│3. P	ACP
	│4. Q • R	1, 3, MP
	│5. R • Q	4, Com
	│6. R	5, Simp
	7. P ⊃ R	3-6, CP
	8. S • T	2, 7, MP
	9. T • S	8, Com
	10. T	9, Simp

QED

Exercises 2.5a. Derive the conclusions of each of the following arguments using the 18 rules and the method of conditional proof.

1. 1. (A ∨ C) ⊃ D
   2. D ⊃ B / A ⊃ B

2. 1. X ⊃ Y
   2. Y ⊃ Z / X ⊃ (Y • Z)

3. 1. Q ⊃ (~R • S) / R ⊃ ~Q

4. 1. ~(P • Q) ⊃ [(~P • ~Q) • (~R • ~S)]
   / P ≡ Q

5. 1. A ⊃ [(D ∨ B) ⊃ C] / A ⊃ (D ⊃ C)

6. 1. Z ⊃ ~Y / (X • Y) ⊃ (Z ⊃ W)

7. 1. ~M ⊃ N
   2. L ⊃ ~N / ~L ∨ M

8. 1. R ⊃ ~O
   2. ~R ⊃ [S • (P ∨ Q)] / O ⊃ (P ∨ Q)

9. 1. ~G ∨ (E • ~F) / (E ⊃ F) ⊃ ~G

10. 1. I ⊃ H
    2. ~I ⊃ J
    3. J ⊃ ~H / J ≡ ~H

11. 1. ~(U ∨ V)
    2. W ⊃ X / (U ∨ W) ⊃ (V ⊃ X)

12. 1. $\sim M \vee N$
    2. P / $(M \vee \sim P) \supset (O \vee N)$

13. 1. $\sim(I \vee \sim K)$
    2. $L \supset J$ / $(I \vee L) \supset (K \bullet J)$

14. 1. $X \supset [(T \vee W) \supset S]$
    2. $(W \supset S) \supset (Y \supset R)$
    3. $\sim Z \supset \sim R$ / $X \supset (Y \supset Z)$

15. 1. $E \supset (F \supset G)$
    2. $\sim(H \vee \sim E)$
    3. $G \supset H$ / $F \supset H$

16. 1. $\sim M \supset \sim(\sim P \vee Q)$
    2. $\sim(O \vee N)$ / $(M \supset O) \supset (P \bullet \sim N)$

17. 1. $(T \supset \sim Q) \bullet \sim W$
    2. $\sim Q \supset [(W \vee S) \bullet (W \vee T)]$
    3. $\sim T \vee (S \supset X)$ / $T \supset X$

18. 1. $[(P \supset (Q \supset P)] \supset S$
    2. $[(P \supset Q) \supset (\sim Q \supset \sim P)] \supset (S \supset T)$ / T

19. 1. $E \supset \sim(F \supset G)$
    2. $F \supset (E \bullet H)$ / $E \equiv F$

20. 1. $M \supset (\sim K \vee N)$
    2. $N \supset L$
    3. $M \vee (K \bullet \sim L)$ / $M \equiv (K \supset L)$

Exercises 2.5b. Translate each of the following paragraphs into arguments written in **PL**. Then, derive the conclusions of the arguments using the 18 rules and the method of conditional proof.

1. If Raul doesn't play lacrosse, then he plays tennis. So, if Raul doesn't play lacrosse, then he plays either tennis or soccer.

2. It is not the case that either Polly or Ramon take out the trash. So, if Owen cleans his room, then Polly takes out the trash only if Quinn clears the table.

3. If Sheldon writes a paper, then using the xerox machine is a necessary condition for Yessenia reading it. Yessenia doesn't read it only if Sheldon doesn't write it. So, if Sheldon writes a paper, the he uses the xerox machine.

4. If Adams and Barnes are translators, then Cooper is a reviewer. Evans is an editor if either Cooper or Durning are reviewers. Hence, Adams being a translator is a sufficient condition for Barnes being a translator only if Evans is an editor.

5. If it's not the case that there are frogs in the pond, then George will go swimming. So if Eloise goes swimming and George does not, then either there are frogs in the pond or hornets in the trees.

6. If Kip does well on his report card, then he will get ice cream. If Kip doesn't do well on his report card, then he'll be jealous of his brother. So, Kip will either get ice cream or be jealous.

7. If Lisa goes to Arizona, then she'll go to Colorado. If she goes Boulder, CO, then she'll go to Dragoon, AZ. So, if she goes to Arizona and Boulder, then she'll go to Colorado and Dragoon.

8. If the train doesn't come, then it is not the case that Shanti and Ricardo go to New York. So, If Ricardo goes to New York, then Shanti goes to New York only if the train comes.

9. Andrew buys Christmas presents only if Clara makes a list. If Clara makes a list, the David will spell check the list for her. Either Clara doesn't make a list or Belinda gets presents. So, if Andrew buys Christmas presents, then Belinda gets presents and David spell checked the list.

10. If Justin goes to Ikea, then Luke doesn't go. Either Luke goes to Ikea or Kate sleeps on the floor. If either Kate or Madeline sleep on the floor, then Justin goes to Ikea. So, Justin goes to Ikea if, and only if, Kate sleeps on the floor.

§2.6: Logical Truths

A theory is a set of sentences, which we call theorems. A formal theory is a set of sentences of a formal language. We identify a theory strictly by its theorems. A logical theory is thus characterized by the set of its logical truths. Logical truths are the theorems of our logical theory, just as certain geometric statements are theorems of Euclidean geometry.

To get a feel for the nature of logical truths, compare 2.6.1, 2.6.2, and 2.6.3.

2.6.1 If it is raining, then I will be unhappy.
2.6.2 If it is raining, then I will get wet.
2.6.3 If it is raining, then it is raining.

Each of the three sentences is expressible in **PL** as ‘P ⊃ Q’. But 2.6.1 and 2.6.2 are contingent sentences. The truth of 2.6.2 is more compelling, but it is still possible for both sentences to be false. 2.6.3, on the other hand, can never be false, as long as we hold the meanings of the terms constant. It is more carefully regimented as ‘P ⊃ P’, and it is a logical truth, or a law of logic.

The logical truths of **PL** are tautologies. We can show, using truth tables, for any wff whether it is a tautology or not. We just look for the wffs which are true in all rows of the truth table. It would be convenient if we also had a method of deriving them within our proof system.

Logical truths do not depend on any premises. They are true no matter what assumptions we make about the world, or whatever we take to be the content of our propositional variables. Thus, we should be able to prove logical truths using any premises, and even without any premises.

One way to derive a theorem with no premises, which we are not using, is to adopt a deductive system which takes certain wffs as axioms. Some theories, including any non-logical theory, are axiomatic. Axiomatic logical theories normally take a few tautologies as axiom schemas. In such a system, any sentence of the form of an axiom can be inserted into a derivation with no further justification.

Our logical theory has no axioms. So far, in order to produce a derivation, we have needed to take some assumptions as premises. We have had no way to construct a derivation with no premises. Now we can use conditional proof to derive logical truths. We can just start our derivation with an assumption, as 2.6.4 does in showing that ‘(P ⊃ Q) ⊃ [(Q ⊃ R) ⊃ (P ⊃ R)]’ is a logical truth.

2.6.4

| 1. P ⊃ Q	ACP
| | 2. Q ⊃ R	ACP
| | 3. P ⊃ R	1, 2, HS
| 4. (Q ⊃ R) ⊃ (P ⊃ R)	2-3, CP
5. (P ⊃ Q) ⊃ [(Q ⊃ R) ⊃ (P ⊃ R)]	1-4, CP

QED

Note that the last line of 2.6.4 is further un-indented than the first line, since the first line is indented. Lines 1-4 are all based on at least one assumption. But, line 5 requires no assumption. It is a theorem of logic, a logical truth.

Derivations of logical truths can look awkward when you are first constructing them. When the logical truth has nested conditionals, as 2.6.4 does, setting up the assumptions can require care. But such logical truths are often simple to derive once they are set up properly. Be careful not to use the assigned proposition in the proof. The conclusion is not part of the derivation until the very end.

2.6.5 shows that ‘[P ⊃ (Q ⊃ R)] ⊃ [(P ⊃ Q) ⊃ (P ⊃ R)]’ is a logical truth.

2.6.5

1. $P \supset (Q \supset R)$	ACP	(to prove $(P \supset Q) \supset (P \supset R)$)
2. $P \supset Q$	ACP	(to prove $(P \supset R)$)
3. P	ACP	(to prove R)
4. $Q \supset R$	1, 3, MP	
5. Q	2, 3, MP	
6. R	4, 5, MP	
7. $P \supset R$	3-6 CP	
8. $(P \supset Q) \supset (P \supset R)$	2-7, CP	
9. $[P \supset (Q \supset R)] \supset [(P \supset Q) \supset (P \supset R)]$	1-8, CP	

QED

Until now, our derivations have required assumptions as premises. Assumptions are usually empirical, taken from observation. Most of the premises of most of the arguments we have seen so far have been contingencies, though we can take any kind of premise, even a contradiction, as an assumption.

Premises of arguments are generally not justified by the same methods which we use to justify our system of logic. Thus, the derivations we have done until this section may be seen as not purely logical. They are not, as they stand, proofs of logical conclusions. They are merely derivations from assumed premises to conclusions. But, for every valid argument requiring premises, we can create a proof of a purely logical truth.

2.6.6

1. $P \supset (Q \bullet R)$
2. $\sim R$ $\quad$ / $\sim P$

The argument 2.6.6 contains two assumptions, at premises 1 and 2. To convert this argument to a logical truth requiring no assumptions, we can construct a conditional statement with the former premises as antecedents and the former conclusion as the consequent. There are two options for how to construct logical truths from any set of premises and a conclusion. On the first option, we conjoin all of the premises into one statement. Then, we take that resulting conjunction as the antecedent of a complex conditional with the conclusion as the consequent. On the second option, we form a series of nested conditionals, using each premises as an antecedent and the conclusion as the final consequent. 2.6.7 shows two the possibilities for turning 2.6.6 into a logical truth.

2.6.7

$\{[P \supset (Q \bullet R)] \bullet \sim R\} \supset \sim P$

$[P \supset (Q \bullet R)] \supset (\sim R \supset \sim P)$

Note the equivalence between the results of the two options by one rule of exportation. Each of the propositions at 2.6.7 is a logical truth, which we show at 2.6.8 using CP.

2.6.8

1. $[P \supset (Q \bullet R)] \bullet \sim R$	ACP
2. $P \supset (Q \bullet R)$	1, Simp
3. $\sim R \bullet [P \supset (Q \bullet R)]$	1, Com
4. $\sim R$	3, Simp
5. $\sim R \vee \sim Q$	4, Add
6. $\sim Q \vee \sim R$	5, Com
7. $\sim (Q \bullet R)$	6, DM
8. $\sim P$	2, 7, MT
9. $\{[P \supset (Q \bullet R)] \bullet \sim R\} \supset \sim P$	1-8, CP

QED

2.6.9 demonstrates the conversion of a longer argument into two logical truths.

2.6.9

1. $P \vee Q$
2. $Q \supset (R \bullet S)$
3. $\sim R$
4. $Q \equiv T$ $\quad / \; P \bullet \sim T$

$$\{\{(P \vee Q) \bullet [Q \supset (R \bullet S)]\} \bullet [\sim R \bullet (Q \equiv T)]\} \supset (P \bullet \sim T)$$

$$(P \vee Q) \supset \{[Q \supset (R \bullet S)] \supset \{\sim R \supset \{(Q \equiv T) \supset (P \bullet \sim T)\}\}\}$$

The relation between derivations requiring assumptions and their corresponding logical truths is guaranteed by a meta-logical result called the Deduction Theorem. The theorem may have been first proved by Alfred Tarski in 1921, but the first published proof was by Jacques Herbrand in 1930. The arguments we have been deriving in this textbook are useful in applying logic to ordinary arguments. But, the logical truths are the logician's real interest, as they are the theorems of propositional logic.

The transformations we have made at the object-language level can also be made at the metalinguistic level. Our rules of inference are written in a metalanguage. Any substitution instances of the premises in our rules of inference entail a substitution instance of the conclusion. We can similarly convert all of our rules of inference. 2.6.10 shows how Modus Ponens can be written as a single sentence of the metalanguage. 2.6.11 shows the same for Constructive Dilemma.

2.6.10

$\alpha \supset \beta$
α $\quad / \; \beta$

can be converted to:

$$[(\alpha \supset \beta) \bullet \alpha] \supset \beta$$

2.6.11

$(\alpha \supset \beta) \bullet (\gamma \supset \delta)$
$\alpha \vee \gamma$ $\quad / \; \beta \vee \delta$

can be converted to:

$$\{[(\alpha \supset \beta) \bullet (\gamma \supset \delta)] \bullet (\alpha \vee \gamma)\} \supset (\beta \vee \delta)$$

Any consistent substitution instance of these new forms, ones in which each metalinguistic variable is replaced by the same wffs of the object language throughout, will be a logical truth.

All ten rules of equivalence we have been using can easily be turned into templates for constructing logical truths even more easily. We can just replace the metalinguistic symbol '$\rightleftarrows$' with the object-language symbol '$\equiv$', as in 2.6.12.

2.6.12

$\sim(\alpha \vee \beta) \equiv (\sim\alpha \bullet \sim\beta)$
$(\alpha \supset \beta) \equiv (\sim\alpha \vee \beta)$

Again, any substitution instance of these forms will be a logical truth.

These metalinguistic templates for logical truths are the kinds of rules one would adopt in an axiomatic system of logic. The templates are called axioms schemas. In an axiomatic theory, we can adopt a small group axiom schemas along with modus ponens and a rule of substitution. Such axiomatic theories can be constructed to derive the same logical theorems, to have the same strength as our system of logic. Again, we are not using an axiomatic system, and so we will retain all eighteen rules, as well as the direct, conditional, and indirect methods of proof, the last of which is the subject of our next section.

Exercises 2.6a. Use conditional proof to derive each of the following logical truths.

1. [A ∨ (B • C)] ⊃ (A ∨ C)

2. [(A ⊃ B) • C] ⊃ (~B ⊃ ~A)

3. (O ∨ P) ⊃ [~(P ∨ Q) ⊃ O]

4. [V • (W ∨ X)] ⊃ (~X ⊃ W)

5. [(D ⊃ ~E) • (F ⊃ E)] ⊃ [D ⊃ (~F ∨ G)]

6. [(H ⊃ I) ⊃ ~(I ∨ ~J)] ⊃ (~H ⊃ J)

7. [(W ⊃ X) • (Y ∨ ~X)] ⊃ [~(Z ∨ Y) ⊃ ~W]

8. [(R • S) ⊃ U] ⊃ {~U ⊃ [R ⊃ (S ⊃ T)]}

9. [(~K ⊃ N) • ~(N ∨ L)] ⊃ [(K ⊃ L) ⊃ M]

10. [(D • E) ⊃ (F ∨ G)] ≡ [(~F • ~G) ⊃ (~D ∨ ~E)]

Exercises 2.6b. Convert each of the following arguments to a logical truth.

1.
1. ~A ⊃ B
2. ~B / A

2.
1. ~ C ∨ D
2. C / D

3.
1. E • (F ∨ G)
2. ~E / G

4.
1. ~(H ∨ I)
2. J ⊃ I / ~J

5.
1. K • (~L ∨ M)
2. L ⊃ ~K / M

6.
1. N ⊃ (P • Q)
2. ~(O ∨ P) / ~N

7.
1. R ⊃ S
2. S ⊃ T
3. ~(T ∨ U) / ~R

8.
1. V ⊃ W
2. ~W ∨ X
3. V • (Y • Z) / X

9.
1. A ∨ (B • C)
2. A ⊃ D
3. ~(D ∨ E) / C

10.
1. F ⊃ G
2. H ⊃ F
3. H • I / ~G ⊃ I

§2.7: Indirect Proof

We have seen two methods of proof, now, the direct and conditional. Lastly for our proof system we have a method of indirect proof. We can see the justification for indirect proof by considering the arguments 2.7.1 and 2.7.2.

2.7.1	1. A • ~A	/ B
	2. A	1, Simp
	3. A ∨ B	2, Add
	4. ~A • A	1, Com
	5. ~A	4, Simp
	6. B	3, 5, DS
QED		

2.7.2	1. B ⊃ (P • ~P)	/ ~B
	| 2. B	ACP
	| 3. P • ~P	1, 2, MP
	| 4. P	3, Simp
	| 5. P ∨ ~B	4, Add
	| 6. ~P • P	3, Com
	| 7. ~P	6, Simp
	| 8. ~B	5, 7, DS
	9. B ⊃ ~B	2-8, CP
	10. ~B ∨ ~B	9, Impl
	11. ~B	10, Taut
QED		

We can infer an interesting moral from each of these arguments. The moral of 2.7.1, which is an instance of what logicians call explosion, is that anything follows from a contradiction. The moral of 2.7.2 is that if a statement entails a contradiction, then its negation is true. Indirect proof is based on these two morals.

Indirect proof is also called *reductio ad absurdum*, or just *reductio*. To use an indirect proof, assume your desired conclusion is false, and try to get a contradiction. If you get it, then you know the negation of your assumption is true.

Method for Indirect Proof

1. Indent, assuming the opposite of what you want to conclude
2. Derive a contradiction, using any wff.
3. Discharge the negation of your assumption.

The last line of an indented sequence for indirect proof is always a contradiction. A contradiction is any statement of the form α • ~α. The wffs listed in 2.7.3 are all contradictions.

2.7.3	P •~ P
	~~P • ~~~P
	~(P∨ ~Q) • ~~(P ∨ ~Q)

We can assume any wff we want in a derivation. But, only certain assumptions will discharge in the desired way. For CP, we assume the antecedent of a desired conditional because when we discharge, the first line of the assumption becomes the antecedent of the resulting conditional. For IP, we always discharge the first line of the proof with one more tilde. Thus, if we wish to prove the negation of a formula, it is perfectly legitimate to assume the formula itself.

2.7.4 is a sample derivation using IP. At line 3, we are considering what would follow if the opposite of the conclusion is true. At line 6, we have found a contradiction, and so we discharge our assumption at line 7.

2.7.4	1. A ⊃ B	
	2. A ⊃ ~B	/~A
	| 3. A	AIP
	| 4. B	1, 3, MP
	| 5. ~B	2, 3, MP
	| 6. B • ~B	4, 5, Conj
	7. ~A	3-6, IP
	QED	

Since the discharge step of an indirect proof requires an extra ~, we often need to use DN at the end of an indirect proof, as in 2.7.5.

2.7.5	1. F ⊃ ~D	
	2. D	
	3. (D • ~E) ⊃ F	/ E
	| 4. ~E	AIP
	| 5. D • ~E	2, 4, Conj
	| 6. F	3, 5, MP
	| 7. ~D	1, 6, MP
	| 8. D • ~D	2, 7, Conj
	9. ~~E	4-8, CP
	10. E	9, DN
	QED	

In addition to deriving simple statements and negations, the method of indirect proof is especially useful for proving disjunctions, as in 2.7.6 Assuming the negation of a disjunction leads quickly, by DM, to two conjuncts that you can simplify.

2.7.6	1. ~A ⊃ (B ⊃ C)		
	2. C ⊃ D		
	3. B	/ A ∨ D	
	│ 4. ~(A ∨ D)		AIP
	│ 5. ~A • ~D		4, DM
	│ 6. ~A		5, Simp
	│ 7. B ⊃ C		1, 6, MP
	│ 8. ~D • ~A		5, Com
	│ 9. ~D		8, Simp
	│ 10. ~C		2, 9, MT
	│ 11. C		7, 3, MP
	│ 12. C • ~C		11, 10, Conj
	13. ~~(A ∨ D)		4-12, IP
	14. A ∨ D		13, DN
QED			

Indirect proof is compatible with conditional proof. Indeed, the fundamental structure of many proofs in mathematics involves making some assumptions and then assuming the opposite of what you want to prove in order to yield a contradiction. 2.7.7 is a formal example of exactly this procedure.

2.7.7	1. E ⊃ (A • D)		
	2. B ⊃ E	/ (E ∨ B) ⊃ A	
	│ 3. E ∨ B		ACP
	│ │ 4. ~A		AIP
	│ │ 5. ~A ∨ ~D		4, Add
	│ │ 6. ~(A • D)		5, DM
	│ │ 7. ~E		1, 6, MT
	│ │ 8. B		3, 7, DS
	│ │ 9. ~B		2, 7, MT
	│ │ 10. B • ~B		8, 9, Conj
	│ 11. ~~A		4-10, IP
	│ 12. A		11, DN
	12. (E ∨ B) ⊃ A	3-12, CP	
QED			

Like conditional proof, the method of indirect proof is easily adapted to proving logical truths. To prove that '~[(X ≡ Y) • ~(X ∨ ~Y)]' is a logical truth, as in 2.7.8, we again start with an assumption, the opposite of the theorem we wish to prove.

2.7.8

	1. $(X \equiv Y) \bullet {\sim}(X \vee {\sim}Y)$	AIP
	2. $X \equiv Y$	1, Simp
	3. $(X \supset Y) \bullet (Y \supset X)$	2, Equiv
	4. ${\sim}(X \vee {\sim}Y)$	1, Com, Simp
	5. ${\sim}X \bullet Y$	4, DM DN
	6. $Y \supset X$	3, Com, Simp
	7. ${\sim}X$	5, Simp
	8. ${\sim}Y$	6, 7, MT
	9. Y	5, Com, Simp
	10. $Y \bullet {\sim}Y$	9, 8, Conj
11. ${\sim}[(X \equiv Y) \bullet {\sim}(X \vee {\sim}Y)]$		1-10, IP

QED

2.7.9 is another example of using IP to derive a logical truth, '$(P \supset Q) \vee ({\sim}Q \supset P)$'.

2.7.9

	1. ${\sim}[(P \supset Q) \vee ({\sim}Q \supset P)]$	AIP
	2. ${\sim}(P \supset Q) \bullet {\sim}({\sim}Q \supset P)$	1, DM
	3. ${\sim}(P \supset Q)$	2, Simp
	4. ${\sim}({\sim}P \vee Q)$	3, Impl
	5. $P \bullet {\sim}Q$	4, DM, DN
	6. ${\sim}({\sim}Q \supset P)$	2, Com, Simp
	7. ${\sim}(Q \vee P)$	6, Impl, DN
	8. ${\sim}Q \bullet {\sim}P$	7, DM
	9. P	5, Simp
	10. ${\sim}P$	8, Com, Simp
	11. $P \bullet {\sim}P$	9, 10, Conj
12. $(P \supset Q) \vee ({\sim}Q \supset P)$		1-11, IP

QED

Here are some hints to help determine whether to use conditional proof or indirect proof to derive a logical truth.

> If the main operator is a conditional or a biconditional, generally use conditional proof.
> If the main operator is a disjunction or a negation, generally use indirect proof.
> If the main operator is a conjunction, we look to the main operators of each conjunct to determine the best method of proof.

We can nest proofs of logical truths inside a larger proof, as intermediate steps, as in 2.7.10. Notice that the antecedents of the conditionals on lines 4 and 10 are logical truths.

2.7.10		
	1. $B \supset [(D \supset D) \supset E]$	
	2. $E \supset \{[F \supset (G \supset F)] \supset (H \bullet {\sim}H)\}$	/ ${\sim}B$
	$\vert$ 3. B	AIP
	$\vert$ 4. $(D \supset D) \supset E$	1, 3, MP
	$\vert\quad\vert$ 5. D	ACP
	$\vert$ 6. $D \supset D$	5, CP
	$\vert$ 7. E	4, 6, MP
	$\vert$ 8. $[F \supset (G \supset F)] \supset (H \bullet {\sim}H)$	2, 7, MP
	$\vert\quad\vert$ 9. F	ACP
	$\vert\quad\vert$ 10. $F \vee {\sim}G$	9, Add
	$\vert\quad\vert$ 11. ${\sim}G \vee F$	10, Com
	$\vert\quad\vert$ 12. $G \supset F$	11, Impl
	$\vert$ 13. $F \supset (G \supset F)$	9-12, CP
	$\vert$ 14. $H \bullet {\sim}H$	8, 13, MP
	15. ${\sim}B$	3-14, IP
QED		

Exercises 2.7a. Use any of the eighteen rules, conditional proof and/or indirect proof to derive the conclusions of the following arguments.

1.
   1. $U \supset (V \vee W)$
   2. ${\sim}(W \vee V)$ / ${\sim}U$

2.
   1. $Y \vee {\sim}Z$
   2. $Z \bullet ({\sim}X \vee W)$ / $X \supset Y$

3.
   1. $X \supset T$
   2. $Y \supset T$
   3. $T \supset Z$ / $(X \vee Y) \supset Z$

4.
   1. $A \supset B$
   2. ${\sim}(C \vee {\sim}A)$ / B

5.
   1. $S \supset T$
   2. $S \vee ({\sim}R \bullet U)$ / $R \supset T$

6.
   1. $F \supset (E \vee D)$
   2. ${\sim}E \bullet ({\sim}D \vee {\sim}F)$ / ${\sim}F$

7.
   1. ${\sim}(K \bullet J)$
   2. $I \vee (L \bullet J)$ / ${\sim}K \vee I$

8.
   1. $X \supset (W \supset Z)$
   2. $Y \vee W$ / ${\sim}Y \supset (X \supset Z)$

9.
   1. $M \supset L$
   2. ${\sim}(K \bullet N) \supset (M \vee L)$ / $K \vee L$

10. 1. $A \equiv (B \bullet D)$
2. $C \supset (E \vee F)$
3. $(A \vee \sim E) \bullet (A \vee \sim F)$ / $C \supset B$

11. 1. $H \supset G$
2. $H \vee J$
3. $\sim(J \vee \sim I)$ / $G \bullet I$

12. 1. $X \supset Y$
2. $\sim(Z \supset W)$ / $X \supset (Y \bullet Z)$

13. 1. $M \supset (L \bullet \sim P)$
2. $K \supset \sim(O \bullet \sim P)$
3. $N \supset \sim O$ / $(K \bullet M) \supset \sim N$

14. 1. $A \supset B$
2. $\sim C \supset \sim(A \vee \sim D)$
3. $\sim D \vee (B \bullet C)$ / $A \supset (B \bullet C)$

15. 1. $P \equiv (Q \vee \sim R)$
2. $T \bullet \sim(Q \bullet P)$ / $\sim(P \bullet R)$

16. 1. $A \equiv \sim(B \vee C)$
2. $(D \vee E) \supset \sim C$
3. $\sim(A \bullet D)$ / $D \supset B$

17. 1. $U \supset (P \bullet \sim Q)$
2. $T \supset (S \vee U)$
3. $\sim T \supset \sim R$ / $(P \supset Q) \supset (R \supset S)$

18. 1. $B \supset C$
2. $E \equiv \sim(B \vee A)$
3. $D \supset \sim E$ / $D \supset (A \vee C)$

19. 1. $Z \supset Y$
2. $Z \vee W$
3. $Y \supset \sim W$
4. $W \equiv \sim X$ / $X \equiv Y$

20. 1. $F \supset (K \equiv M)$
2. $\sim F \supset [L \supset (F \equiv H)]$
3. $\sim(M \vee \sim L)$
4. $\sim H \supset \sim(\sim K \bullet L)$ / $F \equiv H$

Exercises 2.7b. Translate each of the following paragraphs into arguments written in **PL**. Then, derive the conclusions of the arguments using any of the eighteen rules, conditional proof and/or indirect proof.

1. If Lorena makes quiche, then she'll make potatoes. She either doesn't make potatoes or doesn't make quiche. So, she doesn't make quiche.

2. Stephanie either plays miniature golf and not netball, or she goes to the ocean. She doesn't play miniature golf. So, she goes to the ocean.

3. If Grady eats quickly, then he'll get hiccups. If he gets hiccups, then he'll suck on an ice cube and will not eat quickly. So Grady doesn't eat quickly.

4. Julian goes fishing only if Kevin wakes him up. Julian goes fishing alone unless Liam and Julian go fishing together. So, Kevin wakes Julian up.

5. If either Xander or Yael go to the water park, then Vivian will go. Winston going to the water park is sufficient for Vivian not to go. So, If Winston goes to the water park, then Xander will not.

6. If Esme grows olives, then she grows mangoes. She grows either olives or nectarines. So, she grows either mangoes or nectarines.

7. Having gorillas at the circus entails that there are elephants. There are either gorillas or hippos. Having fancy ponies means that there are no hippos. Thus either there are elephants or there are no fancy ponies.

8. Owen will be happier if and only if he either practices the cello or quits music lessons. It is not that case that if Owen quits music lessons then he'll be happier. So, it is not the case that he both never sleeps and practices.

9. If the house is painted ivory and not green, then it will appear friendly. The neighbors are either happy or jealous. If the neighbors are jealous, then the house will be painted ivory. So, if it is not the case that either the house appears friendly or it is painted green, then the neighbors will be happy.

10. If tanks tops are worn is school, then the rules are not enforced. It is not the case that either short skirts or very high heels are in the dress code. Tank tops are worn in school, and either uniforms are taken into consideration or the rules are not enforced. So, it is not the case that either the rules are enforced or shirt skirts are in the dress code.

Appendix: Proofs of the Logical Equivalence of Eight Rules of Equivalence

De Morgan's Rules: $\sim(\alpha \vee \beta) \rightleftarrows \sim\alpha \bullet \sim\beta$

~	(α	∨	β)
0	1	1	1
0	1	1	0
0	0	1	1
1	0	0	0

~	α	•	~	β
0	1	**0**	0	1
0	1	**0**	1	0
1	0	**0**	0	1
1	0	**1**	1	0

De Morgan's Rules: $\sim(\alpha \bullet \beta) \rightleftarrows \sim\alpha \vee \sim\beta$

~	(α	•	β)
0	1	1	1
1	1	0	0
1	0	0	1
1	0	0	0

~	α	∨	~	β
0	1	**0**	0	1
0	1	**1**	1	0
1	0	**1**	0	1
1	0	**1**	1	0

Association: $\alpha \vee (\beta \vee \gamma) \rightleftarrows (\alpha \vee \beta) \vee \gamma$

α	∨	(β	∨	γ)
1	**1**	1	1	1
1	**1**	1	1	0
1	**1**	0	1	1
1	**1**	0	0	0
0	**1**	1	1	1
0	**1**	1	1	0
0	**1**	0	1	1
0	**0**	0	0	0

(α	∨	β)	∨	γ
1	1	1	**1**	1
1	1	1	**1**	0
1	1	0	**1**	1
1	1	0	**1**	0
0	1	1	**1**	1
0	1	1	**1**	0
0	0	0	**1**	1
0	0	0	**0**	0

Association: $\alpha \bullet (\beta \bullet \gamma) \rightleftarrows (\alpha \bullet \beta) \bullet \gamma$

α	•	(β	•	γ)
1	**1**	1	1	1
1	**0**	1	0	0
1	**0**	0	0	1
1	**0**	0	0	0
0	**0**	1	1	1
0	**0**	1	0	0
0	**0**	0	0	1
0	**0**	0	0	0

(α	•	β)	•	γ
1	1	1	**1**	1
1	1	1	**0**	0
1	0	0	**0**	1
1	0	0	**0**	0
0	0	1	**0**	1
0	0	1	**0**	0
0	0	0	**0**	1
0	0	0	**0**	0

Distribution: $\alpha \vee (\beta \bullet \gamma) \rightleftarrows (\alpha \vee \beta) \bullet (\alpha \vee \gamma)$

α	∨	(β	•	γ)
1	**1**	1	1	1
1	**1**	1	0	0
1	**1**	0	0	1
1	**1**	0	0	0
0	**1**	1	1	1
0	**0**	1	0	0
0	**0**	0	0	1
0	**0**	0	0	0

(α	∨	β)	•	(α	∨	γ)
1	1	1	**1**	1	1	1
1	1	1	**1**	1	1	0
1	1	0	**1**	1	1	1
1	1	0	**1**	1	1	0
0	1	1	**1**	0	1	1
0	1	1	**0**	0	0	0
0	0	0	**0**	0	1	1
0	0	0	**0**	0	0	0

Distribution: $\alpha \bullet (\beta \vee \gamma) \rightleftarrows (\alpha \bullet \beta) \vee (\alpha \bullet \gamma)$

α	•	(β	∨	γ)
1	**1**	1	1	1
1	**1**	1	1	0
1	**1**	0	1	1
1	**0**	0	0	0
0	**0**	1	1	1
0	**0**	1	1	0
0	**0**	0	1	1
0	**0**	0	0	0

(α	•	β)	∨	(α	•	γ)
1	1	1	**1**	1	1	1
1	1	1	**1**	1	0	0
1	0	0	**1**	1	1	1
1	0	0	**0**	1	0	0
0	0	1	**0**	0	0	1
0	0	1	**0**	0	0	0
0	0	0	**0**	0	0	1
0	0	0	**0**	0	0	0

Contraposition: $\alpha \supset \beta \rightleftarrows {\sim}\beta \supset {\sim}\alpha$

α	⊃	β
1	**1**	1
1	**0**	0
0	**1**	1
0	**1**	0

~	β	⊃	~	α
0	1	**1**	0	1
1	0	**0**	0	1
0	1	**1**	1	0
1	0	**1**	1	0

Material Implication: $\alpha \supset \beta \rightleftarrows {\sim}\alpha \vee \beta$

α	⊃	β
1	**1**	1
1	**0**	0
0	**1**	1
0	**1**	0

~	α	∨	β
0	1	**1**	1
0	1	**0**	0
1	0	**1**	1
1	0	**1**	0

Material Equivalence: $\alpha \equiv \beta \rightleftarrows (\alpha \supset \beta) \bullet (\beta \supset \alpha)$

α	≡	β
1	**1**	1
1	**0**	0
0	**0**	1
0	**1**	0

(α	⊃	β)	•	(β	⊃	α)
1	1	1	**1**	1	1	1
1	0	0	**0**	0	1	1
0	1	1	**0**	1	0	0
0	1	0	**1**	0	1	0

Material Equivalence: $\alpha \equiv \beta \rightleftarrows (\alpha \bullet \beta) \vee ({\sim}\alpha \bullet {\sim}\beta)$

α	≡	β
1	**1**	1
1	**0**	0
0	**0**	1
0	**1**	0

(α	•	β)	∨	(~	α	•	~	β)
1	1	1	**1**	0	1	0	0	1
1	0	0	**0**	0	1	0	1	0
0	0	1	**0**	1	0	0	0	1
0	0	0	**1**	1	0	1	1	0

Exportation: $(\alpha \bullet \beta) \supset \gamma \rightleftarrows \alpha \supset (\beta \supset \gamma)$

(α	•	β)	⊃	γ
1	1	1	**1**	1
1	1	1	**0**	0
1	0	0	**1**	1
1	0	0	**1**	0
0	0	1	**1**	1
0	0	1	**1**	0
0	0	0	**1**	1
0	0	0	**1**	0

α	⊃	(β	⊃	γ)
1	**1**	1	1	1
1	**0**	1	0	0
1	**1**	0	1	1
1	**1**	0	1	0
0	**1**	1	1	1
0	**1**	1	0	0
0	**1**	0	1	1
0	**1**	0	1	0

Tautology: $\alpha \rightleftarrows \alpha \vee \alpha$

α	∨	α
1	**1**	1
0	**0**	0

Tautology: $\alpha \rightleftarrows \alpha \bullet \alpha$

α	•	α
1	**1**	1
0	**0**	0

Chapter 3: Predicate Logic
§3.1: Translation

We started our study of logic with a casual understanding of what follows from what. Intuitively, a valid argument is one in which the truth of the premises ensures the truth of the conclusion. Then, we explored a semantic definition of validity, in Chapter 1, and a proof system based on that semantic definition, in Chapter 2. Our formal notion of validity for propositional logic captures many intuitively-valid inferences. But it does not capture all of them. For example, argument 3.1.1 is intuitively valid.

3.1.1	All philosophers are happy.
	Emily is a philosopher.
	So, Emily is happy.

But our test for logical validity in propositional logic is of no help.

3.1.2	P	
	Q	/ R

The rules for validity for propositional logic are thus insufficient as a general characterization of logical consequence. **PL** captures entailments among propositions. The entailments in 3.1.1 are within the simple propositions. Thus, we need a logic that explores logical relations inside propositions, not merely those between propositions. Quantificational, or predicate, logic does just that.

In **PL**, we use the following vocabulary.

Capital English letters for simple statements
Five connectives
Punctuation (brackets)

In predicate logic, we extend the vocabulary. We retain the same connectives and punctuation. But, the terms are more complex, revealing some sub-propositional content.

Complex statements made of objects and predicates
Quantifiers
Five connectives
Punctuation

For propositional logic, we used one language, **PL**, and one system of inference. For predicate logic, we will explore several different languages and proof systems. We'll start this section with a general characterization of predicate logic. Then we will proceed to focus on a simple language, which I will call **M**, for monadic predicate logic.

Objects and Predicates

In all predicate logic, we represent objects using lower-case letters.

a, b, c,...u	stand for specific objects and are called constants.
v, w, x, y, z	are used as variables.

In **M**, we represent properties of objects using capital letters, called predicates. Predicates are placed in front of object letters. 'Pa' is used to say that object a has property P, and is said "P of a." A predicate of **M** followed by a constant is called a closed sentence. 3.1.3 shows some closed sentences.

3.1.3	Alicia is clever.	Ca
	Baruch plays chess.	Pb
	Carlos is tall.	Tc

A predicate followed by a variable is called an open sentence. 3.1.4 shows some open sentences. Notice that closed sentences express what we might call a complete proposition whereas open sentences do not. Indeed, they are not easily expressed in English.

3.1.4	v is admirable	Av
	w is bold	Bw
	x is courteous	Cx

The predicates used in 3.1.3 and 3.1.4, and generally in **M**, are called one-place predicates since they are followed by only one object. In §3.8, we will extend our uses of predicates, using capital letters also to stand for relations among various objects.

Returning to 3.1.1, we can now regiment the second premise and the conclusion.

Pe Emily is a philosopher
He Emily is happy

To finish regimenting the argument, to regiment the first premise, we need a quantifier.

Quantifiers

The subject of 'All philosophers are happy' is not a specific philosopher. No specific object is mentioned. Similarly in, 'Something is made in the USA', there is no specific thing to which the sentence refers. For sentences like these, we use quantifiers. There are two kinds of quantifiers: existential and universal. We will use five existential quantifiers.

$(\exists x)$, $(\exists y)$, $(\exists z)$, $(\exists w)$, $(\exists v)$

Existential quantifiers are used to represent expressions like the following.

There exists a thing such that
For some thing
There is a thing
For at least one thing
Something

We will also use five, parallel universal quantifiers.

$(\forall x)$, $(\forall y)$, $(\forall z)$, $(\forall w)$, $(\forall v)$

Universal quantifiers are used to represent expressions like the following.

For all x
Everything

Some terms, like 'anything', can indicate either an existential or universal quantifier, depending on the context.

3.1.5	If anything is missing, you'll be sorry.
3.1.6	Anything goes.

In 3.1.5, we use an existential quantifier. 'Anything' in that case indicates 'something': If something is missing, then you'll be sorry. In 3.1.6, we use a universal quantifier, since that sentence expresses that everything is acceptable. To know whether to use an existential or universal quantifier in cases where a quantifier is called for, you will have to judge from the context of the use.

Here are some examples of simple translations using quantifiers.

3.1.7	Something is made in the USA	$(\exists x)Ux$		
3.1.8	Everything is made in the USA	$(\forall x)Ux$		
3.1.9	Nothing is made in the USA	$(\forall x)\sim Ux$	or	$\sim(\exists x)Ux$

Notice that statements with quantifiers and negations, like 3.1.9, can be translated in at least two different ways. We can say that everything lacks a property or that it is not the case that something has the property.

Most English sentences are best translated using at least two predicates. Very roughly, we can consider many sentences to contain grammatical subjects and grammatical predicates. The grammatical subject is what the sentence is about. The grammatical predicate is what the sentence says about its grammatical subject. For example, in 'Mind-body materialists are chauvinists', the grammatical subject is 'mind-body materialists' and the grammatical predicate is 'are chauvinists'.

When regimenting such sentences, it is typical to use one or more predicate for the grammatical subject of the sentence and another predicate (or more than one) for the grammatical predicate of the sentence. Between the two predicates, there will be a connective. Universally-quantified propositions tend to use conditionals as the operator between the two predicates. Existentially-quantified propositions usually use conjunctions between the two predicates. But these are not absolute rules.

3.1.10	All persons are mortal.	$(\forall x)(Px \supset Mx)$		
3.1.11	Some actors are vain.	$(\exists x)(Ax \bullet Vx)$		
3.1.12	Some gods aren't mortal.	$(\exists x)(Gx \bullet \sim Mx)$		
3.1.13	No frogs are people.	$(\forall x)(Fx \supset \sim Px)$	or	$\sim(\exists x)(Fx \bullet Px)$

Propositions with more than two predicates

The standard, quantified sentence in **M** will have a grammatical subject and a grammatical predicate separated by either a conjunction or a conditional. But many grammatical subjects and predicates will themselves be complex. 3.1.14 and 3.1.15 have more than one predicate in the grammatical-subject portion of the proposition.

3.1.14	Some wooden desks are uncomfortable.	$(\exists x)[(Wx \bullet Dx) \bullet \sim Cx]$
3.1.15	All wooden desks are uncomfortable.	$(\forall x)[(Wx \bullet Dx) \supset \sim Cx]$

3.1.16 and 3.1.17 have more than one predicate in the grammatical-predicate part of the proposition.

3.1.16	Many applicants are untrained or inexperienced.	$(\exists x)[Ax \bullet (\sim Tx \vee \sim Ex)]$
3.1.17	All applicants are untrained or inexperienced.	$(\forall x)[Ax \supset (\sim Tx \vee \sim Ex)]$

When regimenting into predicate logic, start by asking whether the sentence is universal or existential. Then, it is often helpful to think of a sentence in terms of the ordinary rules of subject-predicate grammar. What are we talking about? That's the grammatical-subject portion of the proposition. What are we saying about it? That's the grammatical-predicate portion. The 'what we are talking about' goes as the antecedent in the universally quantified statement, and as the first conjunct in the existentially quantified statement. The 'what we are saying about it' goes as the consequent or as the second conjunct.

Only

'Only' usually indicates a universal quantifier, but translations using 'only' can be tricky.

3.1.18 Only men have been presidents.

3.1.18 claims that if something has been a president, it must have been a man; all presidents have been men. Thus, it should be equivalent to 3.1.19

3.1.19 All presidents have been men.

In other words, 'only Ps are Qs' is logically equivalent to 'all Qs are Ps'. Thus, in some simple cases, we can just invert the antecedent and consequent of a parallel sentence that uses 'all'. Start with the related 'all' sentence, like 3.1.20. Then take the converse to find the 'only' sentence.

3.1.20	All men have been presidents.	$(\forall x)(Mx \supset Px)$
3.1.21	Only men have been presidents.	$(\forall x)(Px \supset Mx)$

When sentences get more complex, the rule of just switching antecedent and consequent between an 'all' sentence and its correlated 'only' sentence has to be adjusted. 3.1.22 is standardly regimented as 3.1.23.

3.1.22 All intelligent students understand Kant.

3.1.23 $(\forall x)[(Ix \bullet Sx) \supset Ux]$

But, if we regiment 3.1.24 merely by taking the converse of the conditional in 3.1.23, we get 3.1.25.

3.1.24 Only intelligent students understand Kant

3.1.25 $(\forall x)[Ux \supset (Ix \bullet Sx)]$

3.1.25 says that anyone who understands Kant must be an intelligent student. It follows from that regimentation that I don't understand Kant, since I am no longer a student. Now, I am not sure whether I understand Kant, but whether I do is not a logical consequence of 3.1.24.

Ordinarily, the preferred regimentation of 3.1.24 is 3.1.26, which says that any student who understands Kant is intelligent.

3.1.26 $(\forall x)[(Ux \bullet Sx) \supset Ix)]$

3.1.26 is a reasonable thing to say. When regimenting, we need not assume that everything that is said is reasonable; that's surely a false assumption. But, it is customary and charitable to presume reasonableness unless we have good reason not to.

I said that to regiment sentences into predicate logic, we think of them as divided into grammatical-subject and grammatical-predicate. In universally-quantified sentences, there is a horseshoe between the grammatical-subject portion or the proposition and the grammatical-predicate portion. In existential sentences, we use a conjunction. In sentences like 3.1.22, the grammatical-subject portion of the sentence has both a subordinate subject ('x is a student') and a subordinate attribute ('x is intelligent'); there is a single predicated attribute ('x understands Kant'). The relation between the only-quantified sentence and its corresponding all-quantified sentence is that the subordinate attribute is switched with the main attribute, but the subordinate subject remains where it is, in the antecedent. Thus, an amended rule could be that if an only-quantified sentence uses only two predicates, you can just switch the antecedent and consequent from the sentence which results from replacing 'all' for 'only'; but if it contains a subordinate subject and attribute in the subject portion of the sentence, then you should just switch the two attributes ('x is intelligent' and 'x understands Kant'), leaving the subordinate subject alone.

3.1.27 'Only PQs are R' is ordinarily the same as 'All RQs are P'

3.1.27 is a good general rule, almost always applicable. But, there are exceptions, and some sentences may be ambiguous. It is not especially clear whether 3.1.28 is best regimented as 3.1.29 or as 3.1.30.

3.1.28 Only famous men have been presidents.
3.1.29 $(\forall x)[(Px \supset (Mx \bullet Fx)]$
3.1.30 $(\forall x)[(Px \bullet Mx) \supset Fx]$

3.1.29 and 3.1.30 are not logically equivalent. 3.1.29 says that if something is a president, then it is a famous man. 3.1.30 says that if something is a male president, then it is famous. Imagine a situation in which there are both men and women presidents. Of the women presidents, some have been famous, and some have been obscure. But, all of the men who have been president have been famous. In such a case, we would favor the second regimentation. But, if we take 'president' to refer to presidents of the United States, the former regimentation is better. Extra-logical information, and not the grammar of the sentence, may favor 3.1.29.

Still, we could imagine a case in which 3.1.30 is the intended interpretation. Consider if we had some men presidents and some women presidents, but the men have all been famous and some of the women have been obscure. Then, we might use 3.1.29, with a bit of an inflection on 'men', to say that of the male presidents, all of them have been famous, but of the women, some have been famous and some have not. 3.1.31 is a good exception to the rule 3.1.27; my thanks to Marianne Janack for the example. Since one must hold a ticket to win the lottery, 'winners of the lottery who are ticket-holders' is redundant. The better regimentation is 3.1.32.

3.1.31 Only probability-challenged ticket-holders win the lottery.
3.1.32 $(\forall x)[Wx \supset (Px \bullet Tx)]$

Propositions with more than one quantifier

Some propositions will contain more than one quantifer. The main operator of such sentences can be any of the four binary connectives, or the negation.

3.1.33 If anything is damaged, then everyone in the house complains.
$(\exists x)Dx \supset (\forall x)[(Ix \bullet Px) \supset Cx]$
3.1.34 Either all the gears are broken, or a cylinder is missing.
$(\forall x)(Gx \supset Bx) \vee (\exists x)(Cx \bullet Mx)$
3.1.35 Some philosophers are realists, while other philosophers are fictionalists.
$(\exists x)(Px \bullet Rx) \bullet (\exists x)(Px \bullet Fx)$
3.1.36 It's not the case that all conventionalists are logical empiricists if and only if some holists are conventionalists.
$\sim[(\forall x)(Cx \supset Lx) \equiv (\forall x)(Hx \supset Cx)]$

Exercises 3.1a. Translate each sentence into predicate logic.

1. Anderson is tall.
2. Belinda sings well.
3. Deanna drives to New York city.
4. The Getty Museum is located in Los Angeles.
5. Snowy is called Milou in Belgium.
6. Cortez and Guillermo go to the gym after school.
7. Either Hilda makes dinner or Ian does.
8. Jenna doesn't run for class president.
9. Ken doesn't walk to school when it rains.
10. Either Lauren or Megan buys lunch.
11. Nate and Orlando play in the college orchestra.
12. Paco will play football only if he's not injured.
13. Ramona plays volleyball if, and only if, she sets up the net.
14. If Salvador invests all his money in the stock market, then he takes a second job.
15. Hamilton College is closed if and only if President Stewart invokes the closure policy.

Exercises 3.1b. Translate each sentence into predicate logic.

1. All computers are difficult to program. (Cx, Dx)
2. All mammals feed their young. (Mx, Fx)
3. Some trees are green. (Tx, Gx)
4. Some flowers do not bloom. (Fx, Bx)
5. Some cherries are red. (Cx, Rx)
6. Every fruit has seeds. (Fx, Sx)
7. A few people walk fast. (Px, Wx)
8. Not all buses are yellow. (Bx, Yx)
9. A cloud is not fluffy. (Cx, Fx)
10. Every mistake is a lesson. (Mx, Lx)
11. Some blankets are not soft. (Bx, Sx)
12. Nothing worthwhile is easy. (Wx, Ex)
13. Most planes are safe. (Px, Sx)
14. Some doctors are not smart. (Dx, Sx)
15. All humans have a mother. (Hx, Mx)
16. Some mountains are not difficult to climb. (Mx, Dx)
17. Not all snakes are poisonous. (Sx, Px)
18. Some spiders are not harmful. (Sx, Hx)
19. No dog has antennae. (Dx, Ax)
20. No lions are not carnivorous. (Lx, Cx)

Exercises 3.1c. Translate each sentence into predicate logic.

1. Some pink flowers are fragrant. (Px, Fx, Sx)
2. Some pink flowers are not fragrant.
3. All red flowers are fragrant. (Rx, Fx, Sx)
4. No orange flowers are fragrant. (Ox, Fx, Sx)
5. Some people are intelligent but not friendly. (Px, Ix, Fx)
6. All friendly people succeed. (Fx, Px, Sx)
7. No friendly people commit crimes. (Fx, Px, Cx)
8. All cats and dogs have whiskers. (Cx, Dx, Wx)
9. Some cats and all dogs have pointed ears. (Cx, Dx, Ex)
10. Not all dogs and cats like humans. (Dx, Cx, Lx)
11. Some brown rats are kept as pets. (Bx, Rx, Kx)
12. No brown rats are used in experiments. (Bx, Rx, Ux)
13. Not all brown rats are dirty. (Bx, Rx, Dx)
14. Some American politicians are immoral. (Ax, Px, Ix)
15. A few American politicians went to ivy league colleges. (Ax, Px, Cx)
16. All presidents are American politicians. (Px, Ax, Lx)
17. Some talented athletes don't receive scholarships. (Tx, Ax, Sx)
18. All talented athletes work hard. (Tx, Ax, Wx)
19. Some athletes have talent if and only if they have determination. (Ax, Tx, Dx)

20. Only talented athletes play professional basketball. (Tx, Ax, Px)
21. Some prime numbers are even. (Px, Nx, Ex)
22. Not all prime numbers are odd. (Px, Nx, Ox)
23. If all prime numbers are odd, then no prime numbers are even. (Px, Nx, Ox, Ex)
24. Only scientists work in labs. (Sx, Lx)
25. All scientists who work in labs have graduated college. (Sx, Lx, Gx)
26. Goats and cows produce milk. (Gx, Cx, Mx)
27. All goats and some cows have horns. (Gx, Cx, Hx)
28. Only short poems rhyme. (Sx, Px, Rx)
29. All successful poets are either creative or hard-working. (Sx, Px, Cx, Hx)
30. Some successful poets are creative but not imaginative. (Sx, Px, Cx, Ix)
31. Either everyone plays water polo or some people go to the concert. (Px, Wx, Cx)
32. Only water polo players are good swimmers. (Wx, Sx)
33. Everyone plays water polo only if they can swim. (Px, Wx, Sx)
34. No one plays soccer unless there is a referee. (Px, Sx, Rx)
35. If any coaches play soccer and some players are interested, then some players will play on a team. (Cx, Sx, Px, Ix, Tx)
36. All safe horses are calm. (Sx, Hx, Cx)
37. Only safe horses are calm.
38. Only calm horses are safe.
39. All undercooked chicken can give you food poisoning. (Ux, Cx, Px)
40. Only undercooked chicken can give you food poisoning.
41. Not all undercooked food can give you food poisoning. (Ux, Fx, Px)
42. Not all extreme sports are dangerous. (Ex, Sx, Dx)
43. Only sports which are not extreme are not dangerous.
44. Every student is healthy unless some student is sick. (Sx, Hx, Ix)
45. A horse is calm if, and only if, it has been well-trained and has a good life. (Hx, Cx, Tx, Gx)
46. Bees and wasps make honey and have stingers. (Bx, Wx, Hx, Sx)
47. If some people have allergies, then some companies make money. (Px, Ax, Cx, Mx)
48. Spiders are poisonous only if they have venom and large fangs. (Sx, Px, Vx, Fx)
49. Some spiders are poisonous and most people are scared of spiders. (Sx, Dx, Px, Fx)
50. Some black rats and all white rats are used in experiments. (Bx, Rx, Wx, Ux)
51. If any teachers are boring and some students are lazy, then some students will not graduate. (Tx, Bx, Sx, Lx, Gx)
52. No jacket is warm unless it has been lined. (Jx, Wx, Lx)
53. A class is productive if, and only if, the material is challenging and students do not procrastinate. (Cx, Px, Mx, Wx)

§3.2: A Family of Predicate Logics

We are starting our study of predicate logic by considering a simple language: monadic predicate logic, or **M**. Predicate logic is monadic if the predicates take only one object. §3.1 - §3.7 focus nearly exclusively on **M**.

When predicates take more than one object, we call them relational and we call the resulting language full first-order predicate logic, or **F**. We use **F** (or some further extension of **F**) from §3.8 to the end of the chapter.

In constructing a formal language, we first specify the language, and then rules for wffs. When we studied propositional logic, we dove right into the full version of the language and its formation rules. Then we used the language in a derivation system. For predicate logic, we proceed more cautiously. Each time we extend the logic, we will generate a slightly new language, with slightly new formation rules. Here are the names of each of the formal languages in this book:

PL: Propositional Logic
M: Monadic First-Order Predicate logic
F: Full First-Order Predicate logic
FF: Full First-Order Predicate logic with functors
S: Second-Order Predicate logic

Another difference between our study of propositional logic and our study of predicate logic is that in this chapter on predicate logic, we will see the difference between a logical language and a system of deduction. In propositional logic, we used one language and one set of inference rules. The language and the deductive system are distinct. We can use the same language in different deductive systems and we can use the same deductive system with different languages. We will use **M** and **F** with the same deductive system. Then, we will add new inference rules covering a special identity predicate. It is typical to name both the deductive systems and the languages, but we need not do so. I will name only the different languages. Let's proceed to the formal characterization of **M**.

Vocabulary of M

Capital letters A...Z used as one-place predicates
Lower case letters used as singular terms
    a, b, c,...u are used as constants.
    v, w, x, y, z are used as variables.
Five connectives: $\sim$, $\bullet$, $\vee$, $\supset$ $\equiv$
Quantifier symbols: $\exists$, $\forall$
Punctuation: (), [], {}

Constants and variables are called **singular terms**. Some first-order systems allow propositional variables like the ones we used in **PL**. We could allow capital letters with no constants or variables directly following them to be propositional variables, as if they were zero-place predicates. But we will not use them.

In order to use quantifers properly, one has to be sensitive to their scope. The quantifiers in 3.2.1 and 3.2.2 have different scope.

3.2.1	$(\forall x)(Px \supset Qx)$	Every P is Q
3.2.2	$(\forall x)Px \supset Qx$	If everything is P, then x is Q

We have already tacitly seen the notion of scope in using negations.

The **scope of a negation** (in **PL**) is whatever directly follows the tilde.

If what follows the tilde is a single propositional variable, then the scope of the negation is just that propositional variable.
If what follows the tilde is another tilde, then the scope of the first (outside) negation is the scope of the second (inside) negation plus that inside tilde.
If what follows the tilde is a bracket, then the entire formula which occurs between the opening and closing of that bracket is in the scope of the negation.

3.2.3 $\sim\{(P \bullet Q) \supset [\sim R \vee \sim\sim(S \equiv T)]\}$

There are four tildes in 3.2.3. The first one has broadest scope. Since what follows it is a bracket, the rest of the formula, everything enclosed in the squiggly brackets, is in the scope of the leading negation. The second tilde in the formula, which occurs just in front of the 'R', has narrow scope. It applies only to the 'R'. The third tilde in the formula has '$\sim(S \equiv T)$' in its scope. The fourth tilde has '$(S \equiv T)$' in its scope.

Similarly, the **scope of a quantifier** is whatever formula immediately follows the quantifier.

If what follows the quantifier is a bracket, then any formulas that occur until that bracket is closed are in the scope of the quantifier.
If what follows the quantifier is a tilde, then the tilde and every formula in its scope is in the scope of the quantifier.
If what follows the quantifier is another quantifier, then the inside quantifier and every formula in the scope of the inside quantifier is in the scope of the outside quantifier.

3.2.4 $(\forall w)\{Pw \supset (\exists x)(\forall y)[(Px \bullet Py) \supset (\exists z)\sim(Qz \vee Rz)]\}$

There are four quantifiers in 3.2.4. Their scopes are as follows.

Quantifier	Scope
$(\forall w)$	$\{Pw \supset (\exists x)(\forall y)[(Px \bullet Py) \supset (\exists z)\sim(Qz \vee Rz)]\}$
$(\exists x)$	$(\forall y)[(Px \bullet Py) \supset (\exists z)\sim(Qz \vee Rz)]$
$(\forall y)$	$[(Px \bullet Py) \supset (\exists z)\sim(Qz \vee Rz)]$
$(\exists z)$	$\sim(Qz \vee Rz)$

Scope is important for quantifiers because it affects which variables are bound by the quantifier. When we construct derivations in predicate logic, we will often remove quantifiers from formulas. When we do so, the variables bound by those quantifiers will be affected. Similarly, we will add quantifiers to front of formulas. When we do so, we bind variables that are in their scopes.

Quantifiers bind every instance of their variable in their scope. A **bound variable** is attached to the quantifier which binds it. In 3.2.1, the 'x' in 'Qx' is bound, as is the 'x' in 'Px'. In 3.2.2, the 'x' in 'Qx' is not bound, though the 'x' in 'Px' is bound. An unbound variable is called a **free variable**.

Wffs that contain at least one unbound variable are called **open sentences**. 3.2.5-3.2.8 are all open sentences.

3.2.5	Ax
3.2.6	$(\forall x)Px \vee Qx$
3.2.7	$(\exists x)(Px \vee Qy)$
3.2.8	$(\forall x)(Px \supset Qx) \supset Rz$

3.2.6, 3.2.7, and 3.2.8 contain both bound and free variables. In 3.2.6, 'Qx' is not in the scope of the quantifier, so is unbound. In 3.2.7, 'Qy' is in the scope of the quantifier, but 'y' is not the quantifier variable, so is unbound. In 3.2.8, 'Rz' is neither in the scope of the quantifier, nor does it contain the quantifier variable.

If a wff has no free variables, it is a **closed sentence**, and expresses a **proposition**. 3.2.9 and 3.2.10 are closed sentences. Translations from English into **M** should ordinarily yield closed sentences.

3.2.9	$(\forall y)[(Py \bullet Qy) \supset (Ra \vee Sa)]$
3.2.10	$(\exists x)(Px \bullet Qx) \vee (\forall y)(Ay \supset By)$

Formation rules for wffs of M

1. A predicate (capital letter) followed by a singular term (lower-case letter) is a wff.
2. For any variable β, if α is a wff that does not contain either '$(\exists\beta)$' or '$(\forall\beta)$', then '$(\exists\beta)\alpha$' and '$(\forall\beta)\alpha$' are wffs.
3. If α is a wff, so is $\sim\alpha$.
4. If α and β are wffs, then so are:
   $(\alpha \bullet \beta)$
   $(\alpha \vee \beta)$
   $(\alpha \supset \beta)$
   $(\alpha \equiv \beta)$
5. These are the only ways to make wffs.

A few observations concerning the formation rules are in order. First, by convention, we drop the outermost brackets which are required by Rule 4. Still, those brackets are implicit and replaced if we augment the formula.

A wff constructed using only Rule 1 is called an **atomic formula**. 3.2.11 - 3.2.13 are atomic formulas. Notice that an atomic formula can be closed (as in 3.2.11 and 3.2.12) or open (as in 3.2.13).

3.2.11	Pa
3.2.12	Qt
3.2.13	Ax

A wff that is part of another wff is called a **subformula**. The proposition 3.2.14 has all of the formulas in the list 3.2.15 as subformulas.

3.2.14 (Pa • Qb) ⊃ (∃x)Rx

3.2.15 Pa
Qb
Rx
(∃x)Rx
Pa • Qb

Quantifiers, like connectives, are called **operators**. Atomic formulas lack operators. The last operator added according to the formation rules is called the **main operator**.

Lastly on the formation rules, Rule 2 contains a clause used to prevent overlapping quantifiers. This clause prevents us from constructing propositions like the ill-formed 3.2.16.

3.2.16 (∃x)[Px • (∀x)(Qx ⊃ Rx)]

The terms ‘Qx’ and ‘Rx’ contain variables that appear to be bound by both the leading existential quantifier and the universal quantifier inside the proposition. In the first few sections of Chapter 3, we won’t normally be tempted to construct such sentences. But after we introduce relational predicates, we will have to be very careful to avoid such overlapping.

We are using only a small, finite stock of singular terms and quantifiers. It is customary to use a larger stock, in fact an infinite stock. To generate an indefinite number ofsingular terms and quantifiers, we could use the indexing functions of subscripts and superscripts. We introduce arabic numerals, say, into the language. Then, we index each constant and variable:

a_1, a_2, a_3...
x_1, x_2, x_3...

We can create an indefinite number of quantifiers by using the indexed variables. More austere languages avoid introducing numbers, and just use prime symbols.

a', a'', a''', a''''...
x', x'', x''', x''''....

Both of these techniques quickly become uinwieldy. Since we are only going to need a few variables and constants, we can use a cleaner, if more limited, syntax, remembering that there is a technique to extend our stock if we were to need it.

Lastly, the language of **M** may appear to be arbitrarily restrictive. When translating into **PL**, I urged that we should reveal as much logical structure as we can. **F** will reveal more logical structure than **M**. Let’s take a short moment to see how.

3.2.17 Andrés loves Beatriz

Using **M**, we read 3.2.17 as predicating a property (loving Beatriz) of an individual (Andrés). We regiment it as 3.2.18.

3.2.18 La

In contrast, if we want to use 'L' to stand for the more-general relation of one thing loving a second, it will have to take two objects: the lover and the lovee. Thus, we could, in **F**, introduce a relational predicate, 'Lxy', which stands for 'x loves y', yielding 3.2.19.

3.2.19 Lab

Relational predicates will allow us greater generality in our translation, just as translating negative statements using tildes did. We look to reveal as much logical structure as we can. But, relational predicates will come later, as an extension of **M**. We will extend our logical language in other ways too, including a specific predicate for identity, and adding functors and second-order quantifiers.

We start with **M**, rather than the more-general **F**, for two reasons. First, there are metalogical differences between **M** and **F**. **M** is decidable, which means that there is a decision procedure, an algorithm, for determining whether or not an inference or theorem is logically valid. **F** is undecidable, which means that determining whether an inference or theorem is valid may require ingenuity.

Second, starting with the language **M**, since it is simpler than **F**, will allow you to become comfortable with the methods of translation and inference before reaching the complications of full, first-order logic.

Exercises 3.2. For each of the following wffs of **M**, answer each of the following questions:

A. For each quantifier in the sentence, which subformulas are in its scope? (List them all.)
B. For each quantifier in the sentence, which variables are bound by the quantifier?
C. Which variables in the sentence are free?
D. Is the sentence open or closed?
E. What is the main operator of the sentence?

1. (∃x)(Px • Qx)

2. (∀x)[(Px • Qx) ⊃ ~Ra]

3. (∀x)(Px • Qx) ⊃ (∃x)[(Px ∨ Qy) ∨ Rx]

4. (∃x)Py

5. (∀x)Px ⊃ (Qx • Ra)

6. ~(∀x)[Px ∨ (~Qy • Rx)]

7. (∀y)(Pa ⊃ Qb)

8. (∃x)(Ry • Qx) • Pa

9. (∃x)(Rx • ~Qx) ≡ (∀x)(Px ⊃ Qa)

10. (Pa ∨ Qb) ⊃ Rc

11. (∀x)(Px ∨ Qx) ⊃ (∀y)(~Qy ⊃ ~Py)

12. (∃x){[(Px ∨ Rx) • Qy] ⊃ (∀y)[(Rx ⊃ Qy) • Pb]}

13. ~(∀x)[(Px ≡ Rx) ⊃ Qa]

14. ~(∃y)(Qx ∨ Px)

15. (∀x){(Px • Qy) ⊃ (∃y)[(Ry ⊃ Sy) • Tx]}

§3.3: Derivations in **M**

We will use a single deductive system with **M**. All of the eighteen rules we used with **PL** continue to hold. In addition, there are four new rules governing removing and adding quantifiers which we will see in this section. The general structure of many of the derivations we will do in this section is first to take off quantifiers; second, to use the rules we already saw for **PL**; and, last, to put quantifiers on. In the next section, we will add an additional rule for exchanging the universal and existential quantifiers.

Taking off the universal quantifier

Recall the valid argument 3.1.1, which we can now fully regiment.

3.1.1	All philosophers are happy.	$(\forall x)(Px \supset Hx)$	
	Emily is a philosopher.	Pe	
	So, Emily is happy.	He	

We need a rule that will generate the conclusion out of the premises. We need to remove the quantifier in such a way as to make the conclusion follow as a simple matter of modus ponens.

Rule #1: Universal Instantiation (UI)

$$\frac{(\forall\alpha)\mathscr{F}\alpha}{\mathscr{F}\beta}$$

for any variable α, any formula $\mathscr{F}$, and any singular term β

To use UI, we remove the leading universal quantifier. Then, we replace all occurrences of variables bound by that quantifier with either a variable (v, w, x, y, z) or a constant (a, b, c,...u). We can use UI only when the main operator of a formula is a quantifier. In other words, UI, like the rules of inference in Chapter 2, may be used only on whole lines.

UI is also governed by a binding rule. When you use instantiate or generalize, you must change all the bound variables in the same way. Thus, 3.3.1 can be instantiated as any formula in the list 3.3.2

3.3.1	$(\forall x)[Sx \vee (Pa \bullet Tx)]$
3.3.2	$Sa \vee (Pa \bullet Ta)$
	$Sb \vee (Pa \bullet Tb)$
	$Sx \vee (Pa \bullet Tx)$
	$Sy \vee (Pa \bullet Ty)$

But 3.3.1 can not be instantiated as 3.3.3 or as 3.3.4.

3.3.3	$Sa \vee (Pa \bullet Tb)$
3.3.4	$Sx \vee (Pa \bullet Ta)$

Let's see how we use UI in the derivation of the argument at 3.1.1.

3.3.5	1. $(\forall x)(Px \supset Hx)$	
	2. Pe	/ He
	3. $Pe \supset He$	1, UI
	4. He	3, 2, MP

QED

Putting on the universal quantifier

All of the propositions in 3.3.6 contain quantifiers as main operators. To derive the conclusion, we will remove the quantifiers, make some inferences, and put on a quantifier at the end.

3.3.6	1. Everything happy is content.	1. $(\forall x)(Hx \supset Cx)$
	2. No miser is content.	2. $(\forall x)(Mx \supset \sim Cx)$
	So, no miser is happy.	/ $(\forall x)(Mx \supset \sim Hx)$

We thus need a rule allowing us to put a quantifier on the front of a formula. We might be tempted to introduce a rule such as 3.3.7.

3.3.7 Bad Generalization Rule

$$\frac{\mathscr{F}a}{(\forall x)\mathscr{F}x}$$

To see why 3.3.7 is a bad generalization rule, consider the instance of it at 3.3.8.

3.3.8
1. Pa
2. $(\forall x)Px$

Now, interpret 'P' as 'is portly' and 'a' as 'Adam'. 3.3.7 thus licenses the conclusion that everything is portly from just one instance. Such an inference is called the Fallacy of Hasty Generalization. To avoid such a fallacy, we never universally generalize (or quantify) over a constant. In other words, we may not replace a constant with a variable bound by a universal quantifier. This restriction keeps us from ever universally quantifying over individual cases.

While we may not quantify over constants, we may quantify over variables. Indeed, the point of introducing variables, and distinguishing them from constants, is precisely to identify when a universal generalization is permitted. Variables, except in limited circumstances we will introduce in §3.5, retain a universal character, even when they are unbound. Generalizing over them (i.e. binding them with a universal quantifier) does not commit a fallacy because the variable can stand for anything and everything.

Rule #2: Universal Generalization (UG)

$$\frac{\mathscr{F}\beta}{(\forall\alpha)\mathscr{F}\alpha}$$ for any variable β, any formula $\mathscr{F}$, and any variable α

Again, UG works only on whole lines. We place the universal quantifier in front of a whole statement so that the scope of the quantifier is the entire rest of the proposition. Further, we replace all occurrences of the variable over which we are quantifying with the variable in the quantifier: we bind all instances of the variable. You must replace all occurrences!

3.3.9 contains a proper use of UG. Notice that I changed all the 'x's to 'y's when instantiating at lines 3 and 4. I could have kept the variables as 'x's, or used any other variable. But, if I had instantiated to constants, which would have been permissible according to UI, I could not have generalized to a universal quantifier at line 7.

3.3.9	1. $(\forall x)(Hx \supset Cx)$	
	2. $(\forall x)(Mx \supset {\sim}Cx)$	/ $(\forall x)(Mx \supset {\sim}Hx)$
	3. $Hy \supset Cy$	1, UI
	4. $My \supset {\sim}Cy$	2, UI
	5. ${\sim}Cy \supset {\sim}Hy$	3, Cont
	6. $My \supset {\sim}Hy$	4, 5, HS
	7. $(\forall x)(Mx \supset {\sim}Hx)$	6, UG
QED		

Putting on the existential quantifier

We now have rules for removing and putting on the universal quantifier. There are parallel rules for the existential quantifer. We will use the rule for existentially generalizing to facilitate the inference 3.3.10.

3.3.10	Oscar is a Costa Rican.	Co
	So there are Costa Ricans.	$(\exists x)Cx$

Rule #3: Existential Generalization (EG)

$$\frac{\mathscr{F}\beta}{(\exists\alpha)\mathscr{F}\alpha}$$ for any singular term β, any formula $\mathscr{F}$, and for any variable α

To use EG, place an existential quantifier in front of any proposition and change all occurrences of the singular term (constant or variable) over which you are quantifying with the quantifier letter. Unlike UG, which results in a strong, universal claim, EG is a weak inference and so can be made from any claim, whether concerning constants or variables. Again, the resulting formula will have the quantifer you just added as the main operator.

The derivation of the argument at 3.3.10 is trivial.

3.3.11	1. Co	/ $(\exists x)Cx$
	2. $(\exists x)Cx$	1, EG
QED		

Taking off the existential quantifier

Our fourth rule for managing quantifiers allows us to remove an existential quantifier. As with UG, there will be a restriction.

3.3.12	All New Yorkers are Americans.	1. $(\forall x)(Nx \supset Ax)$
	Some New Yorkers are bald.	2. $(\exists x)(Nx \bullet Bx)$
	So, some Americans are bald.	/ $(\exists x)(Ax \bullet Bx)$

In order to derive 3.3.12, we have to take off the '∃x' in the second premise. The existential quantifier only commits us to the existence of one thing. So, when we take it off, we have to put on a constant. Moreover, we can not have said anything earlier in the derivation about that constant; it has to be a new thing.

Rule #4: Existential Instantiation (EI)

$$\frac{(\exists\alpha)\mathscr{F}\alpha}{\mathscr{F}\beta}$$ for any variable α, any formula $\mathscr{F}$, and any *new* constant β

As with all four of the quantifier management rules, EI must be used only on whole lines. We remove the leading existential quantifier and replace all occurrences which were bound by the quantifier with the same, new constant. A new constant is one that does not appear in either the premises or the desired conclusion. An existentially-quantified sentence only commits you to the existence of some thing that has the property ascribed to it in the formula, and not to any particular thing which might have other properties inconsistent with those in the formula.

To see why a new constant is required, consider what would happen without that restriction, in the fallacious inference at 3.3.13.

3.3.13	1. (∃x)(Ax • Cx)	
	2. (∃x)(Ax • Dx)	
	3. Aa • Ca	1, EI
	4. Aa • Da	2, EI: but wrong!
	5. Ca	3, Com, Simp
	6. Da	4, Com, Simp
	7. Ca • Da	5, 6, Conj
	8. (∃x)(Cx • Dx)	7, EG

Uh-oh!

To see that 3.3.13 contains a fallacious inference, let's interpret 'Ax' as 'x is an animal'; 'Cx' as 'x is a cat' and 'Dx' as 'x is a dog'. The first two premises are perfectly reasonable: there are cats and there are dogs. The conclusion indicates the existence of a cat-dog. Whatever the advances in biogenetic engineering may be, we can not infer the existence of a cat-dog from the existence of cats and dogs.

Since EI contains a restriction whereas UI does not, in the common case in which you have to instantiate both universally-quantified and existentially-quantified propositions, EI before you UI.

3.3.14 contains an acceptable use of EI.

3.3.14	1. (∀x)(Nx ⊃ Ax)	
	2. (∃x)(Nx • Bx)	/ (∃x)(Ax • Bx)
	3. Na • Ba	2, EI
	4. Na ⊃ Aa	1, UI
	5. Na	3, Simp
	6. Aa	4, 5, MP
	7. Ba	3, Com, Simp
	8. Aa • Ba	6, 7, Conj
	9. (∃x)(Ax • Bx)	8, EG

QED

I mentioned, for each of the rules, that you may not instantiate a line on which the quantifier is not the main operator. As an example, note that in 3.3.15, line 2 can not be instantiated.

3.3.15	1. $(\forall x)(Dx \bullet Ex)$	
	2. $(\forall x)Dx \supset Fa$	/ $(\exists x)Fx$
	3. $Dx \bullet Ex$	1, UI
	4. Dx	3, Simp
	5. $(\forall x)Dx$	4, UG
	6. Fa	2, 5, MP
	7. $(\exists x)Fx$	6, EG
QED		

Similarly, we can not take off either quantifier in line 1 of 3.3.16.

3.3.16	1. $(\forall x)(Jx \vee Kx) \supset (\exists y)Ly$	
	2. $(\forall x)(Jx \vee Lx)$	
	3. $(\forall x)({\sim}Lx \vee Kx)$	/ $(\exists x)Lx$
	4. $Jx \vee Lx$	2, UI
	5. ${\sim}Jx \supset Lx$	4, DN, Impl
	6. ${\sim}Lx \vee Kx$	3, UI
	7. $Lx \supset Kx$	6, Impl
	8. ${\sim}Jx \supset Kx$	5, 7, HS
	9. $Jx \vee Kx$	8, Impl, DN
	10. $(\forall x)(Jx \vee Kx)$	9, UG
	11. $(\exists y)Ly$	1, 10, MP
	12. La	11, EI
	13. $(\exists x)Lx$	12, EG
QED		

You may instantiate the same quantifier twice, including the existential quantifier. When the quantifier is universal, as in 3.3.17, there are no restrictions on instantiating it.

3.3.17	1. $(\forall x)(Mx \supset Nx)$	
	2. $(\forall x)(Nx \supset Ox)$	
	3. $Ma \bullet Mb$	/ $Na \bullet Ob$
	4. $Ma \supset Na$	1, UI
	5. Ma	3, Simp
	6. Na	4, 5, MP
	7. $Mb \supset Nb$	1, UI
	8. Mb	3, Com, Simp
	9. Nb	7, 8, MP
	10. $Nb \supset Ob$	2, UI
	11. Ob	10, 9, MP
	12. $Na \bullet Ob$	6, 11, Conj
QED		

When the quantifier is existential, as in 3.3.18, the second instantiation must go to a new constant. It may seem odd that we can instantiate an existential quantifier twice when the use of an existential quantifier only commits you to a single thing having a given property. But, that odd feeling should be removed by remembering that objects may have more than one name.

3.3.18	1. $(\exists x)(Px \bullet Qx)$		
	2. $(\forall x)(Px \supset Rx)$		/ $(\exists x)Rx \bullet (\exists x)Qx$
	3. $Pa \bullet Qa$	1, EI	
	4. $Pa \supset Ra$	2, UI	
	5. Pa	3, Simp	
	6. Ra	4, 5, MP	
	7. $(\exists x)Rx$	6, EG	
	8. $Pb \bullet Qb$	1, EI	
	9. $Qb \bullet Pb$	8, Com	
	10. Qb	9, Simp	
	11. $(\exists x)Qx$	10, EG	
	12. $(\exists x)Rx \bullet (\exists x)Qx$	7, 11, Conj	

QED

Exercises 3.3. Derive the conclusions of the following arguments.

1.	1. $(\exists y)(Ny \bullet Oy)$	/ $Na \bullet Ob$
2.	1. $(\forall x)Hx \vee Ja$	
	2. $(\forall x)[(\sim Jx \bullet Ix) \vee (\sim Jx \bullet Kx)]$	/ $(\forall x)Hx$
3.	1. $(\exists x)(Px \bullet \sim Qx)$	/ $(\exists x)(Qx \supset Rx)$
4.	1. $(\exists x)(Tx \bullet Ux) \supset (\forall x)Vx$	
	2. $(\exists x)[(Wx \bullet Tx) \bullet Ux]$	/ $(\forall x)Vx$
5.	1. $(\forall x)(Ax \supset Bx)$	
	2. $(\forall x)(Cx \supset \sim Bx)$	
	3. Aa	/ $\sim Ca$
6.	1. $(\forall x)(Ax \supset Bx)$	
	2. $(\forall x)(Cx \supset \sim Bx)$	/ $(\forall x)(Cx \supset \sim Ax)$
7.	1. $(\forall x)(Jx \bullet Kx)$	/ $(\exists x)Jx \bullet (\exists x)Kx$
8.	1. $(\exists x)(Dx \bullet \sim Ex)$	
	2. $(\forall x)(Ex \vee Fx)$	/ $(\exists x)Fx$
9.	1. $(\exists x)(Ax \bullet \sim Bx)$	
	2. $(\forall x)(Cx \supset Bx)$	/ $(\exists x)(Ax \bullet \sim Cx)$
10.	1. $(\exists x)(Px \bullet Qx)$	
	2. $(\exists x)(Rx \bullet Sx)$	/ $Pa \bullet Rb$

11.	1. $(\exists x)(Fx \bullet Hx) \equiv Gb$	
	2. Gb	/ Fa
12.	1. $(\forall x)(Fx \equiv Gx)$	/ $(\forall x)(Fx \supset Gx) \bullet (\forall x)(Gx \supset Fx)$
13.	1. $(\forall x)Ax \supset Ba$	
	2. $(\forall x){\sim}(Ax \supset Cx)$	/ $(\exists x)Bx$
14.	1. $(\exists x)Lx \equiv Nb$	
	2. $(\exists x)[(Lx \bullet Mx) \bullet Ox]$	/ $(\exists x)Nx$
15.	1. $(\forall x)(Fx \vee Hx) \supset (\exists x)Ex$	
	2. $(\forall x)[Fx \vee (Gx \bullet Hx)]$	/ $(\exists y)Ey$
16.	1. $(\forall x)(Ix \supset Kx)$	
	2. $(\forall x)(Jx \supset Lx)$	
	3. $(\exists x)(Jx \vee Ix)$	/ $(\exists x)(Kx \vee Lx)$
17.	1. $(\forall x)[Gx \supset (Hx \vee Ix)]$	
	2. $(\exists x)(Gx \bullet {\sim}Ix)$	/ $(\exists x)(Gx \bullet Hx)$
18.	1. $(\forall x)(Ax \equiv Cx)$	
	2. $(\forall x)(Bx \supset Cx)$	
	3. Ba	/ $(\exists x)Ax$
19	1. $(\exists x)(Ax \bullet Bx)$	
	2. $(\forall x)(Ax \supset Cx)$	
	3. $(\forall x)(Bx \supset Dx)$	/ $(\exists x)(Cx \bullet Dx)$
20.	1. $(\forall x)(Mx \supset Nx)$	
	2. $(\forall x)(Ox \supset Px)$	
	3. $(\forall x)[Mx \vee (Ox \bullet Qx)]$	/ $(\forall x)(Nx \vee Px)$
21.	1. $(\exists x)(Ax \bullet Bx) \vee ({\sim}Ca \bullet Da)$	
	2. $(\forall x)(Dx \supset Cx)$	/ $(\exists x)(Ax \bullet Bx)$
22.	1. $(\forall x)(Fx \bullet Gx)$	
	2. $(\exists x)({\sim}Gx \vee Ex)$	/ $(\exists x)(Fx \bullet Ex)$
23.	1. $(\forall x)(Mx \supset Nx)$	
	2. $(\exists x)({\sim}Nx \bullet Ox)$	
	3. $(\exists x){\sim}Mx \supset (\exists x){\sim}Ox$	/ $(\exists x)Ox \bullet (\exists x){\sim}Ox$
24.	1. $(\forall x)(Dx \supset Ex)$	
	2. $(\forall x)(Ex \supset {\sim}Gx)$	
	3. $(\exists x)Gx$	/ $(\exists x){\sim}Dx$

25. 1. $(\exists x)(Mx \bullet Ox) \supset (\exists x)Nx$
2. $(\exists x)(Px \bullet Mx)$
3. $(\forall x)({\sim}Px \vee Ox)$ / $(\exists x)Nx$

26. 1. $(\forall x)(Dx \bullet Ex)$
2. $(\exists x)({\sim}Fx \vee Gx)$ / $(\exists x)[(Dx \equiv Ex) \bullet (Fx \supset Gx)]$

27. 1. $(\forall x)[Tx \vee (Ux \bullet Vx)]$
2. $(\forall x)(Wx \supset {\sim}Tx)$ / $(\forall x)(Wx \supset Ux)$

28. 1. $(\forall x)(Lx \equiv Nx)$
2. $(\forall x)(Nx \supset Mx)$
3. $(\forall x){\sim}(Mx \vee Ox)$ / $(\exists x){\sim}Lx$

29. 1. $(\exists x)[Hx \bullet (Ix \vee Jx)]$
2. $(\forall x)(Kx \supset {\sim}Ix)$
3. $(\forall x)(Hx \supset Kx)$ / $(\exists x)Jx$

30. 1. $(\exists x)(Dx \bullet Fx)$
2. $(\exists x)(Gx \supset Ex)$
3. $(\forall x){\sim}(Hx \vee Ex)$ / $(\exists x)Fx \bullet (\exists x){\sim}Gx$

31. 1. $(\forall x)(Rx \equiv Tx)$
2. $(\exists x)(Tx \bullet {\sim}Sx)$
3. $(\forall x)\ [Sx \vee (Rx \supset Ux)]$ / $(\exists x)Ux$

32. 1. $(\forall x)Ix \supset (\forall x)Kx$
2. $(\forall x)[Jx \bullet (Ix \vee Lx)]$
3. $(\forall x)(Jx \supset {\sim}Lx)$ / $(\forall x)Kx$

33. 1. $(\forall x)(Px \vee Qx) \equiv Rc$
2. $(\forall x){\sim}(Sx \vee {\sim}Qx)$ / $(\exists y)Ry$

34. 1. $(\exists x)Qx \equiv (\exists x)Sx$
2. $(\forall x)(Rx \vee Sx)$
3. $(\exists x){\sim}(Rx \vee Qx)$ / Qb

35. 1. $(\exists x)Ax \supset (\forall x)Cx$
2. $(\forall x)({\sim}Bx \supset Dx)$
3. $(\forall x)(Bx \supset Ax)$
4. $(\exists x){\sim}(Dx \vee {\sim}Cx)$ / $(\forall x)Cx$

36. 1. $(\forall x)(Kx \supset Lx)$
2. $(\forall x)(Lx \supset Mx)$
3. $Ka \bullet Kb$ / $(\exists x)Lx \bullet (\exists y)My$

37. 1. $(\forall x)(Ox \supset Qx)$
2. $(\forall x)(Ox \vee Px)$
3. $(\exists x)(Nx \bullet {\sim}Qx)$ / $(\exists x)(Nx \bullet Px)$

38. 1. $(\forall x)(Px \supset Qx)$
2. $(\forall x){\sim}(Rx \vee {\sim}Px)$ / $(\exists x)(Qx \bullet {\sim}Rx)$

39. 1. $(\forall x)[Ax \supset (Bx \vee Cx)]$
2. $(\exists x){\sim}(Bx \vee {\sim}Ax)$ / $(\exists x)Cx$

40. 1. $(\exists x)[(Sx \vee Tx) \bullet Ux]$
2. $(\forall x)(Ux \supset {\sim}Sx)$ / $(\exists x){\sim}Sx \bullet (\exists y)(Uy \bullet Ty)$

41. 1. $(\forall x)(Sx \vee Tx)$
2. $(\forall x){\sim}(Ux \supset Sx)$ / $(\forall x)Tx \bullet (\exists y)Uy$

42. 1. $(\forall x)(Hx \supset {\sim}Jx)$
2. $(\forall x)(Ix \supset Jx)$
3. $Ha \bullet Ib$ / ${\sim}(Ia \vee Hb)$

43. 1. $(\exists x)Ax$
2. $(\forall x)(Ax \supset Bx)$
3. $(\forall x){\sim}(Ex \supset Bx)$ / $(\exists x)Cx$

44. 1. $(\exists x)({\sim}Tx \bullet Ux) \equiv (\forall x)Wx$
2. $(\forall x)(Tx \supset Vx)$
3. $(\exists x)(Ux \bullet {\sim}Vx)$ / $(\forall x)Wx$

45. 1. $(\exists x)(Jx \equiv Kx) \supset (\forall x)(Ix \bullet Lx)$
2. $(\forall x)[(Ix \bullet Jx) \supset Kx]$
3. $(\exists x){\sim}(Ix \supset Kx)$ / $(\forall y)Ly$

46. 1. $(\exists x)Kx \supset (\forall x)(Lx \supset Mx)$
2. $(\forall x){\sim}(Kx \supset {\sim}Lx)$
3. $(\forall x){\sim}Mx$ / $(\exists x){\sim}Lx$

47. 1. $(\exists x)[Ix \vee (Hx \vee Jx)]$
2. $(\forall x){\sim}({\sim}Ix \supset Jx)$
3. $(\forall x){\sim}(Hx \bullet Kx)$ / $(\exists x){\sim}Kx$

§3.4: Quantifier Exchange

The rules for removing and replacing quantifiers will allow us to make many inferences in predicate logic. But, some inferences need more machinery. Consider the argument at 3.4.1 and its natural regimentation.

3.4.1 All successful football players are hard-working. But, not all football players are hard working. So, not everything is successful.

1. $(\forall x)[(Fx \bullet Sx) \supset Hx]$
2. $\sim(\forall x)(Fx \supset Hx)$ $\quad$ / $\sim(\forall x)Sx$

W need to remove the quantifer in the second premise of 3.4.1 to derive the conclusion. But, the quantifier is not the main operator of that proposition and so we can not instantiate the premise as it stands. Further, we will want to put a quantifier on some proposition near the end of the derivation. But, it's unclear how we are going to sneak the quantifier in between the tilde and the 'Sx' in the conclusion. We need some rules for managing the interactions between quantifiers and negations. As we will see in this section, every proposition which has a negation in front of a quantifier is equivalent to another proposition in which a quantifier is the main operator.

We use two different quantifiers in predicate logic, the existential and the universal. These quantifiers are inter-definable. Indeed, some systems of logic take only one quantifer as fundamental and introduce the other by definition. We can see the equivalence between the existential and universal quantifiers in natural language by considering the following four pairs.

3.4.2	Everything is made of atoms.
3.4.2'	It's not the case that something is not made of atoms.
3.4.3	Something is made of atoms.
3.4.3'	It's wrong to say that nothing is made of atoms.
3.4.4	Nothing is made of atoms.
3.4.4'	It's false that something is made of atoms.
3.4.5	At least one thing isn't made of atoms.
3.4.5'	Not everything is made of atoms.

Noting the equivalence of each pair, we can show their equivalence in the predicate logic regimentations.

$(\forall x)Ax$	is equivalent to	$\sim(\exists x)\sim Ax$
$(\exists x)Ax$	is equivalent to	$\sim(\forall x)\sim Ax$
$(\forall x)\sim Ax$	is equivalent to	$\sim(\exists x)Ax$
$(\exists x)\sim Ax$	is equivalent to	$\sim(\forall x)Ax$

We can thus introduce the rule of **Quantifier Exchange (QE)**. You may replace any expression of one of the above forms with a statement of its logically equivalent form. Like rules of equivalence, QE is based on logical equivalence, rather than validity, and thus may be used on part of a line.

QE
$$(\forall x)\mathscr{F}x \rightleftarrows \sim(\exists x)\sim\mathscr{F}x$$
$$(\exists x)\mathscr{F}x \rightleftarrows \sim(\forall x)\sim\mathscr{F}x$$
$$(\forall x)\sim\mathscr{F}x \rightleftarrows \sim(\exists x)\mathscr{F}x$$
$$(\exists x)\sim\mathscr{F}x \rightleftarrows \sim(\forall x)\mathscr{F}x$$

QE appears as four rules. But, we call really consider them as one, more-general rule. There are three spaces around each quantifier:

1. Directly before the quantifier
2. The quantifier itself
3. Directly following the quantifier

QE says that to change a quantifier, you change each of the three spaces.

Add or remove a tilde directly before the quantifier.
Switch quantifiers: existential to universal or vice versa.
Add or remove a tilde directly after the quantifier.

Some transformations permitted by QE

Understanding the relation between the existential and universal quantifiers facilitates some natural transformations, like between 3.4.5 and 3.4.6, as the following derivation shows.

3.4.6	It's not the case that every P is Q	$\sim(\forall x)(Px \supset Qx)$
3.4.7	Something is P and not Q	$(\exists x)(Px \bullet \sim Qx)$

1. $\sim(\forall x)(Px \supset Qx)$	Premise
2. $(\exists x)\sim(Px \supset Qx)$	1, QE
3. $(\exists x)\sim(\sim Px \vee Qx)$	2, Impl
4. $(\exists x)(Px \bullet \sim Qx)$	3, DM, DN

Similarly, 3.4.8, 3.4.9, and 3.4.10 are all equivalent.

3.4.8	It's not the case that something is both P and Q	$\sim(\exists x)(Px \bullet Qx)$
3.4.9	Everything that's P is not Q	$(\forall x)(Px \supset \sim Qx)$
3.4.10	Everything that's Q is not P	$(\forall x)(Qx \supset \sim Px)$

1. $\sim(\exists x)(Px \bullet Qx)$	Premise
2. $(\forall x)\sim(Px \bullet Qx)$	1, QE
3. $(\forall x)(\sim Px \vee \sim Qx)$	2, DM
4. $(\forall x)(Px \supset \sim Qx)$	3, Impl
5. $(\forall x)(Qx \supset \sim Px)$	4, Cont, DN

3.4.11 is a sample derivation using QE.

3.4.11	1. $(\exists x)Lx \supset (\exists y)My$	
	2. $(\forall y)\sim My$	/ $\sim La$
	3. $\sim(\exists y)My$	2, QE
	4. $\sim(\exists x)Lx$	1, 3, MT
	5. $(\forall x)\sim Lx$	4, QE
	6. $\sim La$	5, UI

QED

Note that in 3.4.11 you can not existentially instantiate line 4. You may only use EI when it is the main operator on a line. On line 4, the main operator is the tilde. Thus, you must use QE before instantiating.

Let's return to 3.4.1. It does not appear, at first glance, to have an existential premise. But since the main operator at line 2 is a tilde in front of a quantifier, in order to instantiate, we first must use QE, yielding an existential sentence. Then, we EI (line 6) before we UI (line 7).

3.4.12	1. $(\forall x)[(Fx \bullet Sx) \supset Hx]$	
	2. $\sim(\forall x)(Fx \supset Hx)$	/ $\sim(\forall x)Sx$
	3. $(\exists x)\sim(Fx \supset Hx)$	2, QE
	4. $(\exists x)\sim(\sim Fx \vee Hx)$	3, Impl
	5. $(\exists x)(Fx \bullet \sim Hx)$	4, DM, DN
	6. $Fa \bullet \sim Ha$	5, EI
	7. $(Fa \bullet Sa) \supset Ha$	1, UI
	8. $\sim Ha$	6, Com, Simp
	9. $\sim(Fa \bullet Sa)$	7, 8, MT
	10. $\sim Fa \vee \sim Sa$	9, DM
	11. Fa	6, Simp
	12. $\sim Sa$	10, 11, DN, DS
	13. $(\exists x)\sim Sa$	12, EG
	14. $\sim(\forall x)Sx$	13, QE

QED

Most of the proofs we have been doing require some instantiation and/or generalization. Now that we have QE available, we can derive arguments which require no removal or replacement of quantifiers, like 3.4.13.

3.4.13	1. $(\forall x)\sim Dx \supset (\forall x)Ex$	
	2. $(\exists x)\sim Ex$ /$(\exists x)Dx$	
	3. $\sim(\forall x)Ex$	2, QE
	4. $\sim(\forall x)\sim Dx$	1, 3, MT
	5. $(\exists x)Dx$	4, QE

QED

Exercises 3.4. Derive the conclusions of each of the following arguments.

1.	1. $(\forall x)Ax \supset (\exists x)Bx$	
	2. $(\forall x){\sim}Bx$	/ ${\sim}Ab$
2.	1. $(\exists x)[Qx \bullet (Rx \bullet {\sim}Sx)]$	/ ${\sim}(\forall y)Sy$
3.	1. $(\forall x)(Jx \bullet Kx) \vee {\sim}(\forall x)Lx$	
	2. ${\sim}Ja$	/ $(\exists x){\sim}Lx$
4.	1. $(\exists x){\sim}Ix \supset (\forall x)(Jx \vee Kx)$	
	2. ${\sim}(\forall x)Ix \bullet {\sim}Jb$	/ Kb
5.	1. $(\exists x)Cx \vee (\forall x)Dx$	
	2. $(\forall x){\sim}(Cx \vee Ex)$	/ $(\forall x)Dx$
6.	1. ${\sim}(\exists x)(Rx \vee Sx) \vee (\forall x)(Tx \supset {\sim}Rx)$	
	2. Ra	/ ${\sim}(\forall x)Tx$
7.	1. $(\exists x){\sim}Fx \vee (\forall x)(Gx \bullet Hx)$	
	2. $(\forall x)[(Fx \bullet Gx) \vee (Fx \bullet Hx)]$	/ $(\exists y)(Gy \bullet Hy)$
8.	1. ${\sim}(\forall x)(Qx \supset Rx)$	
	2. $(\forall x)({\sim}Rx \supset Tx)$	/ ${\sim}(\forall x){\sim}Tx$
9.	1. $(\forall x)[Lx \vee (Mx \bullet {\sim}Nx)]$	
	2. ${\sim}(\exists x)Lx$	/ ${\sim}(\exists x)(Lx \vee Nx)$
10.	1. $(\forall x)[(Tx \bullet Ux) \supset Vx]$	
	2. ${\sim}(\forall x){\sim}Tx$	/ ${\sim}(\forall x)(Ux \bullet {\sim}Vx)$
11.	1. $(\forall x)(Ax \vee Bx)$	
	2. $(\forall x)(Ax \supset Dx)$	
	3. ${\sim}(\forall x)(Bx \bullet {\sim}Cx)$	/ $(\exists y)(Dy \vee Cy)$
12.	1. ${\sim}(\forall x)[Kx \supset (Lx \supset Mx)]$	
	2. $(\forall x)[(Nx \bullet Ox) \equiv Mx]$	/${\sim}(\forall x)(Nx \bullet Ox)$
13.	1. ${\sim}(\exists x)(Ox \equiv Px)$	
	2. Pa	/${\sim}(\forall x)Ox$
14.	1. ${\sim}(\exists x)[Ex \bullet (Fx \vee Gx)]$	
	2. $(\forall x)[Hx \supset (Ex \bullet Gx)]$	
	3. $(\exists x)[{\sim}Hx \supset (Ix \vee Jx)]$	/ $(\exists x)({\sim}Ix \supset Jx)$
15.	1. ${\sim}(\exists x)[Fx \bullet (Gx \bullet Hx)]$	
	2. ${\sim}(\exists x)(Ix \bullet {\sim}Fx)$	/ $(\forall x)[Ix \supset ({\sim}Gx \vee {\sim}Hx)]$

16. 1. ~(∀x)[(Jx • Kx) • Lx]
 2. (∀x)(Mx ⊃ Jx)
 3. (∀x)(~Nx • Mx) / ~(∀x)(Kx • Lx)

17. 1. (∃x)[(Ax ∨ Cx) ⊃ Bx]
 2. ~(∃x)(Bx ∨ Ex)
 3. (∃x)(Dx ⊃ Ex) ⊃ (∀x)(Ax ∨ Cx) / (∃y)Dy

18. 1. (∃x)(Nx ∨ ~Ox)
 2. ~(∀x)(Px • Qx) • ~(∃x)(Nx ∨ ~Qx) / ~[(∀x)Px ∨ (∀x)Ox]

19. 1. (∃x)(Mx • ~Nx) ⊃ (∀x)(Ox ∨ Px)
 2. ~(∀x)(~Nx ⊃ Ox)
 3. ~(∃x)Px / ~(∀y)Ny

20. 1. (∃x)[Ax • (Bx ∨ Cx)] ⊃ (∀x)Dx
 2. ~(∀x)(Ax ⊃ Dx) / ~(∀x)Cx

21. 1. (∀x)(Ex • Fx) ∨ ~(∀x)[Gx ⊃ (Hx ⊃ Ix)]
 2. ~(∀x)(Jx ⊃ Ex) / ~(∀y) Iy

22. 1.~(∃x)(Jx • ~Kx)
 2. ~(∃x)[Kx • (~Jx ∨ ~Lx)] / (∀x)(Jx ≡ Kx)

23. 1. ~[(∃x)(Ax ∨ Bx) • (∀x)(Cx ⊃ Dx)]
 2. ~(∀x)(~Ax ∨ Ex) / (∃x)Cx

24. 1. (∀x)(Fx ⊃ Hx) ∨ ~(∃x)(Gx ≡ Ix)
 2. (∃x)[Fx • (~Hx • Ix)] / ~(∀x)Gx

25. 1. ~(∃x)[Px • (Qx • Rx)]
 2. ~(∀x)[~Rx ∨ (Sx •Tx)]
 3. (∀x)(Px • Qx) ∨ (∀x)(Tx ⊃ Rx) / ~(∃x)(Tx • ~Rx)

§3.5: Conditional and Indirect Proof in **M**

The conditional and indirect methods of proof work just as well in **M** as they did in **PL**, with one small restriction. The restriction arises from considering the unrestricted and fallacious derivation 3.5.1.

3.5.1	1. (∀x)Rx ⊃ (∀x)Bx	Premise
	| 2. Rx	ACP
	| 3. (∀x)Rx	2, UG: but wrong!
	| 4. (∀x)Bx	1, 3, MP
	| 5. Bx	4, UI
	6. Rx ⊃ Bx	2-5, CP
	7. (∀x)(Rx ⊃ Bx)	6, UG

Uh-oh.

Allowing line 7 to follow from the premise at line 1 would be wrong. We can show that the inference is invalid by interpreting the predicates. Let's take 'Rx' to stand for 'x is red' and 'Bx' to stand for 'x is blue'. 3.5.1 would allow the inference of 'Everything red is blue' (the conclusion) from 'If everything is red, then everything is blue' (the premise). But that premise can be true while the conclusion is false. Indeed, since it is not the case that everything is red, the first premise is vacuously true; it is a conditional with a false antecedent. But, the conclusion is clearly false; it is not the case that all red things are blue. So, the derivation should be invalid. We must restrict conditional proof.

The problem with 3.5.1 can be seen at step 3. The assumption for conditional proof at line 2 just means that a random thing has the property denoted by 'R', not that everything has that property. While variables retain their universal character in a proof, when they are used within an assumption (for CP or IP), they lose that universal character. It is as if we are saying, "Imagine that some (particular) thing has the property ascribed in the assumption." If it follows that the object in the assumption also has other properties, we may universally generalize after we've discharged, as in line 7. For, we have not made any specific claims about the thing outside of the assumption.

Using conditional proof in this way should be familiar to mathematics students. Often in mathematics we will show that some property holds of a particular example. Then, we claim, without loss of generality, that since our example was chosen arbitrarily, our property holds universally. Within the assumption, we have a particular example and we treat it existentially. Once we are done with that portion of the proof, we can treat our object universally, without loss of generality.

Consider an indirect proof of some universally quantified formula, '(∀x)α'. We assume its opposite: '~(∀x)α'. We can then change that assumption, using QE, to '(∃x)~α'. In other words, we start with an existential assertion: let's say that something is not α. Another way to do such a proof would be to assume ~α immediately. We could do this by making the free variables in α constants or variables. Either way, they have to act as constants within the assumption.

To summarize, we may not generalize on a variable within the scope of an assumption in which that variable is free. This restriction holds on both CP and IP, though it is rare to use IP with a free variable in the first line.

Restriction on CP and IP

Never use UG within an assumption on a variable that is free in the first line of the assumption.

There are two typical uses of CP in predicate logic. One involves assuming the antecedent of the conditional we ordinarily find inside a universally-quantified formula, as in 3.5.2.

3.5.2	1. $(\forall x)[Ax \supset (Bx \vee Dx)]$	
	2. $(\forall x){\sim}Bx$	/ $(\forall x)(Ax \supset Dx)$
	3. Ay	ACP
	4. $Ay \supset (By \vee Dy)$	1, UI
	5. $By \vee Dy$	4, 3, MP
	6. ${\sim}By$	2, UI
	7. Dy	5, 6, DS
	8. $Ay \supset Dy$	3-7, CP
	9. $(\forall x)(Ax \supset Dx)$	8, UG
QED		

In 3.5.2, at line 3, we pick a random object that has property A. From lines 3 to 7, we show that given any object, if it has A, then it has D; we make that claim at step 8. Then, at line 9, since we are no longer within the scope of the assumption, we may use UG.

To prove statements of the form $(\forall x)(\alpha x \supset \beta x)$, we use the method sketched at 3.5.3.

3.5.3	Assume αx
	Derive βx
	Discharge $(\alpha x \supset \beta x)$
	UG

The other typical use of CP within predicate logic is even more obvious. When you have a proposition whose main operator is a $\supset$, you assume the whole antecedent to prove the whole consequent, as in 3.5.4.

3.5.4	1. $(\forall x)[Px \supset (Qx \bullet Rx)]$	
	2. $(\forall x)(Rx \supset Sx)$	/ $(\exists x)Px \supset (\exists x)Sx$
	3. $(\exists x)Px$	ACP
	4. Pa	3, EI
	5. $Pa \supset (Qa \bullet Ra)$	1, UI
	6. $Qa \bullet Ra$	5, 4, MP
	7. Ra	6, Com, Simp
	8. $Ra \supset Sa$	2, UI
	9. Sa	8, 7, MP
	10. $(\exists x)\, Sx$	9, EG
	11. $(\exists x)Px \supset (\exists x)Sx$	3-10, CP
QED		

Thus, conditional proof can work just as it did in **PL**. Indirect Proof also works the same way it did in propositional logic, as you can see in 3.5.5. Remember, after you make your assumption, you're looking for any contradiction. A contradiction may be an atomic formula and its negation or it may be a more complex formula and its negation. It can contain quantifiers, or not.

3.5.5	1. $(\forall x)[(Ax \vee Bx) \supset Ex]$	
	2. $(\forall x)[(Ex \vee Dx) \supset {\sim}Ax]$	/ $(\forall x){\sim}Ax$
		3. ${\sim}(\forall x){\sim}Ax$	AIP
		4. $(\exists x)Ax$	3, QE
		5. Aa	4, EI
		6. $(Ea \vee Da) \supset {\sim}Aa$	2, UI
		7. ${\sim}(Ea \vee Da)$	6, 5, DN, MT
		8. ${\sim}Ea \bullet {\sim}Da$	7, DM
		9. ${\sim}Ea$	8, Simp
		10. $(Aa \vee Ba) \supset Ea$	1, UI
		11. ${\sim}(Aa \vee Ba)$	10, 9, MT
		12. ${\sim}Aa \bullet {\sim}Ba$	11, DM
		13. ${\sim}Aa$	12, Simp
		14. $Aa \bullet {\sim}Aa$	5, 13, Conj
	15. $(\forall x){\sim}Ax$	3-13, IP, DN

QED

With CP, sometimes you assume only part of a line, and then generalize outside the assumption. With IP, you almost always assume the negation of the whole conclusion, as in line 3 of 3.5.5.

Exercises 3.5. Derive the conclusions of the following arguments.

1.	1. $(\forall x)(Dx \vee Ex)$	
	2. $(\forall x)(Fx \supset {\sim}Ex)$	/ $(\forall x)({\sim}Dx \supset {\sim}Fx)$
2.	1. $(\forall x)(Ax \supset Bx)$	
	2. $(\forall x){\sim}(Bx \bullet {\sim}Cx)$	/ $(\forall x)(Ax \supset Cx)$
3.	1. $(\forall x)(Gx \supset Hx)$	
	2. ${\sim}(\exists x)(Ix \bullet {\sim}Gx)$	
	3. $(\forall x)({\sim}Hx \supset Ix)$	/ $(\forall x)Hx$
4.	1. $(\forall x)[Ax \supset (Bx \supset Cx)]$	
	2. ${\sim}(\forall x)(Bx \supset Dx)$	/ $(\forall x)Ax \supset (\exists x)(Cx \bullet {\sim}Dx)$
5.	1. $(\forall x)(Rx \supset Ux)$	
	2. ${\sim}(\exists x)(Ux \bullet Sx)$	/ $(\exists x)Rx \supset (\exists x){\sim}Sx$
6.	1. $(\forall x)[Ax \supset (Dx \vee Ex)]$	
	2. $(\forall x)[({\sim}Dx \supset Ex) \supset ({\sim}Cx \supset Bx)]$	/ $(\forall x)[Ax \supset (Bx \vee Cx)]$
7.	1. $(\forall x)[{\sim}Nx \vee (Qx \bullet Rx)]$	
	2. $(\forall x)(Px \equiv Qx)$	/ $(\exists x)Nx \supset (\exists x)Px$
8.	1. $(\forall x)(Px \supset Qx)$	
	2. ${\sim}(\exists x)[(Px \bullet Rx) \bullet Qx]$	
	3. $(\exists x)Rx$	/ ${\sim}(\forall x)Px$
9.	1. $(\forall x)(Ox \supset Nx)$	
	2. $(\forall x)(Nx \supset Px)$	
	3. ${\sim}(\exists x)(Px \vee Qx)$	/ $(\forall x){\sim}Ox$
10.	1. $(\forall x)[(Fx \vee Gx) \supset Ix]$	
	2. $(\forall x)[(Ix \bullet Ex) \supset Gx]$	/ $(\forall x)[Ex \supset (Fx \supset Gx)]$
11.	1. $(\forall x)[Sx \supset ({\sim}Tx \vee {\sim}Rx)]$	
	2. $(\forall x)(Ux \supset Sx)$	/ $(\exists x)(Rx \bullet Tx) \supset (\exists x)({\sim}Sx \bullet {\sim}Ux)$
12.	1. $(\forall x)(Ex \equiv Hx)$	
	2. $(\forall x)(Hx \supset {\sim}Fx)$	/ $(\forall x)Ex \supset {\sim}(\exists x)Fx$
13.	1. $(\forall x)(Cx \supset Ax)$	
	2. $(\exists x){\sim}Bx \supset (\forall x)Cx$	/ $(\exists x)(Ax \vee Bx)$
14.	1. $(\forall x)[Jx \supset ({\sim}Kx \supset {\sim}Lx)$	
	2. $(\exists x)(Jx \bullet {\sim}Kx)$	/ ${\sim}(\forall x)Lx$
15.	1. $(\forall x)[Jx \supset (Mx \bullet Lx)]$	
	2. $(\forall x)[({\sim}Kx \vee Nx) \bullet ({\sim}Kx \vee Lx)]$	/ $(\forall x)[(Jx \vee Kx) \supset Lx]$

16. 1. $(\forall x)(Ix \supset Kx)$
2. $(\forall x)(Lx \supset Jx)$
3. $\sim(\exists x)(\sim Kx \supset Jx)$ / $\sim(\exists x)[Ix \vee (Lx \bullet Mx)]$

17. 1. $(\forall x)(Px \supset Ox)$
2. $(\forall x)(Ox \equiv Qx)$ / $(\forall x)(\sim Px \vee Qx)$

18. 1. $(\forall x)[Fx \supset (Dx \bullet \sim Ex)]$
2. $(\forall x)(Fx \supset Hx)$
3. $(\exists x)Fx$ / $\sim(\forall x)(Dx \supset Ex) \vee (\exists x)[Fx \bullet (Gx \bullet Hx)]$

19. 1. $(\exists x)(Sx \vee Tx)$
2. $(\exists x)(Ux \supset \sim Vx)$
3. $(\exists x)Tx \supset (\forall x)Ux$ / $\sim(\forall x)(\sim Sx \bullet Vx)$

20. 1. $(\forall x)[Ax \supset (Cx \bullet Dx)]$
2. $(\exists x)(Bx \bullet \sim Cx)$ / $\sim(\forall x)(Ax \equiv Bx)$

21. 1. $\sim(\exists x)[Rx \equiv (Tx \bullet Ux)]$
2. $(\forall x)\{(Tx \supset \sim Ux) \supset [Sx \equiv (Rx \vee Wx)]\}$ /$(\forall x)[Rx \supset (Sx \vee Vx)]$

22. 1. $(\forall x)[(Lx \bullet Ix) \supset \sim Kx]$
2. $(\forall x)[Mx \vee (Jx \bullet Nx)]$
3. $(\forall x)(Kx \supset \sim Mx)$
4. $(\exists x)(Ix \bullet Kx)$ / $\sim(\forall x)(Jx \supset Lx)$

23. 1. $(\forall x)(Ax \equiv Dx)$
2. $(\forall x)[(\sim Bx \supset Cx) \supset Dx]$
3. $(\forall x)[(Ex \supset Bx) \bullet (Dx \supset Cx)]$ / $(\forall x)[Ax \equiv (Bx \vee Cx)]$

24. 1. $(\exists x)[Fx \vee (Gx \bullet Hx)]$
2. $(\forall x)[\sim Jx \supset (\sim Fx \bullet \sim Hx)]$
3. $(\forall x)(\sim Gx \supset \sim Jx)$ / $(\exists x)(Fx \vee Gx)$

25. 1. $\sim(\exists x)[(Kx \bullet Lx) \bullet (Mx \equiv Nx)]$
2. $(\forall x)\{Kx \supset [Ox \vee (Px \supset Qx)]\}$
3. $(\forall x)[(Lx \bullet Mx) \supset Px]$
4. $(\forall x)[Nx \vee (Kx \bullet \sim Qx)]$ / $(\forall x)[Lx \supset (Nx \vee Ox)]$

§3.6: Semantics for Predicate Logic

We have been constructing and using formal theories of logic. A *theory* is just a set of sentences, which we call theorems. A formal theory is a set of sentences of a formal language. We identify a theory by its theorems, the set of sentences provable within that theory. Some theories are finite, having finitely many theorems. Many interesting formal theories are infinite. The theories we are using based on **PL** and **M** are infinite since they have infinitely many theorems.

To construct a formal theory, we first specify a language and its syntax: vocabulary and rules for well-formed formulas. We have looked carefully at the syntax of both **PL** and **M**. Once we have specified the wffs of a language, we can use that language in a theory. The same language can be used in different theories.

There are different ways to specify the theorems of a theory. Most obviously, we can list them. Listing the theorems of an infinite theory is an arduous task. Alternatively, we can adopt some axioms and rules of inference. For example, we could adopt the axioms of Euclidean geometry or of Newtonian mechanics. Such theories are usually placed within a background logical theory. Their (proper) axioms are added to the logical axioms. To construct formal physical theories, we generally add mathematical axioms as well. Euclid and Newton were not as careful about their rules of inference or background logic as we are today. Frege's logic and the development of proof theory in the twentieth century were responses to worries about the nature of inference. Frege wanted a gap-free logic, and so specified his rules of inference syntactically. The meta-theoretic study of axioms and rules of inference is called proof theory.

Independent of proof theory, we can also provide a semantics for our language. The study of the semantics of a formal language is called model theory. In semantics, we assign truth values to the simple sentences of the language and truth conditions for the construction of complex sentences. We can determine which wffs are logically true and which inferences are valid by using model theory.

In proof theory, we specify the theorems and acceptable inferences. In model theory, we characterize logical truth and validity. In propositional logic, the theorems were exactly the logical truths. So, proof theory and model theory have the same results for propositional logic.

A formal theory, like our system of propositional logic, is called complete when all the logically true wffs are provable. A theory is called sound when every provable formula is logically true. **PL** is both complete and sound. In more sophisticated theories, proof separates from truth. Kurt Gödel's first incompleteness theorem shows that in theories with just some weak mathematical axioms, there will be true sentences that are not provable. Gödel uses arithmetic to allow a formal theory to state properties like provability within the theory. He constructs a predicate, 'is provable' that holds of sentences only with specific, statable arithmetic properties. Then, he constructs a theorem that says, truly, of itself that it is not provable. Since it is true, it is not provable. Thus, in theories which allow the Gödel construction, model theory and proof theory provide different results.

In **PL**, our semantics consists of constructing truth tables. We simply interpret the sentences of **PL**, by assigning 1 or 0 to each atomic sentence. We compute truth values of complex propositions by combining, according to the truth table definitions, the truth values of the atomic sentences. Since we have only twenty-six simple terms, the capital English letters, there are only $2^{26} = \sim 6.7$ million possible interpretations. That is a large number, but it is a finite number.

A more useful language will have infinitely many simple terms: P, P', P'', P'''... A language with infinitely many formulas will have an even greater infinitely many interpretations. Still, since we are working with only two truth values, we can determine the logical truths even in a language with infinitely many variables. We just look at the truth tables.

In **PL**, our proof system consisted of our eighteen rules of natural deduction. Systems of natural deduction seem to mirror ordinary reasoning. The rules of inference are often intuitive. Despite having no axioms, we were able to prove theorems using indirect and conditional methods.

Other proof systems use axioms. Here is an example of an axiomatic system I'll call **PS** in the language of propositional logic.

Formal system PS

Language and wffs: those of **PL**[1]

Axiom Schemata:

For any wffs α, β, and γ, statements of the following forms are axioms:

AS1: $\alpha \supset (\beta \supset \alpha)$

AS2: $(\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma))$

AS3: $(\sim\alpha \supset \sim\beta) \supset (\beta \supset \alpha)$

Rule of inference:

Modus ponens

PS and our system of natural deduction are provably equivalent, since they are equivalent languages, and both systems are complete. Our system of natural deduction makes proofs shorter and easier than they would be in axiomatic systems of logic. Natural deduction systems have one main drawback: their metalogical proofs are more complicated. When we reason about the system of logic we have chosen, we ordinarily choose a system that is as simple as it can be: few symbols, few rules. If we want to show that a system of natural deduction is legitimate, we can show that it is equivalent to a more austere system.

Logical truth and validity were easy to define in **PL**, using the truth tables. In **M** and the other languages of predicate logic we will study the semantics are more complicated. Remember, separating the syntax of our language from its semantics allows us to treat our formal languages as completely uninterpreted. We can take our proof system as an empty game of manipulating formal symbols.

Intuitively, we know what the logical operators mean. But until we specify a formal interpretation, we are free to interpret them as we wish. Similarly, our constants and predicates and quantifiers are, as far as the syntax of our language specifies, uninterpreted. To look at the logical properties of the language, we interpret the logical particles variously. This way, we can see what is essential to the language itself and what is imposed by an interpretation.

Interpretations

The first step in formal semantics for predicate logic is to show how to provide an interpretation of a language. Then, we can determine its logical truths. The logical truths will be the wffs that come out as true under every interpretation.

To define an interpretation in **M**, or in any of its extensions, we have to specify how to handle constants, predicates and quantifiers. To interpret predicates and quantifiers, we use some set theory. We need not add set theory to our object language, but we need it in our metalanguage. We interpret a first-order theory in four steps.

Step 1. Specify a set to serve as a domain of interpretation.

The domain of interpretation will be the universe of the theory, the objects to which we are applying the theory. (Sometimes it is called a domain of quantification.) We can consider small finite domains, like a universe of three objects: $U_1 = \{1, 2, 3\}$; or $U_2 = \{$Barack Obama, Hillary Clinton, and Rahm Emanuel$\}$. Or, we can consider larger domains, like a universe of everything.

---

[1] We do not need any of the wffs which use ∨, •, and ≡; see §4.5.

Actually, there is no set of everything because such a set would lead to paradox. The set of woodchucks, for example, is not too large to be a set. But, the set of things which are not woodchucks is too large. Among the things which are not woodchucks are sets. If we take a set to be any collection, among the sets would be the set of all sets which are not members of themselves. But that seemingly-well-defined set is paradoxical. If it belongs to itself, then it can not belong to itself. If it does not belong to itself, then it should.

This paradox, which Bertrand Russell found in Frege's set theory, shows that not every property, like the property of not being a woodchuck, determines a set. In such cases, we can consider the collection a proper class instead of a set. One must be careful handling proper classes, since they are explosive. But, we will not run into difficulties with them here.

Step 2. Assign a member of the domain to each constant.

We introduced constants to be used as names of particular things. In giving an interpretation of our language, we pick one thing out of the domain for each constant. Different constants may correspond to the same object, just as an individual person or thing can have multiple names.

For example, if we were using **M** and working with a small domain of interpretation $U_1 = \{1, 2, 3\}$, we could assign the number one to 'a', the number two to 'b', and the number three to all of the remaining nineteen constants ('c'...'u').

Step 3. Assign some set of objects in the domain to each predicate.

We interpret predicates as sets of objects in the domain of which that predicate holds. If we use a predicate 'Ex' to stand for 'x has been elected president', then the interpretation of that predicate will be the set of things that were elected president. In U_1, the interpretation of 'Ex' will be empty; in U_2 it will be {Barack Obama}. We can interpret predicates by providing a list of members of the domain or by providing a rule. In the domain of natural numbers, for instance, we might define a predicate of even numbers, say, as $\{x \mid x = 2n, \text{ for n in the domain}\}$.

Step 4. Use the customary truth tables for the interpretation of the connectives.

We are familiar with this part of the semantics from **PL**.

In order to characterize truth for sentences of a formal theory of predicate logic, we first define satisfaction. Then we can define truth for an interpretation.

Objects in the domain may satisfy predicates; ordered n-tuples (pairs, triples, quadruples, etc.) may satisfy relations. A wff will be satisfiable if there are objects in the domain of quantification which satisfy the predicates indicated in the wff. A universally quantified sentence is satisfied if it is satisfied by all objects in the domain. An existentially quantified sentence is satisfied if it is satisfied by some object in the domain.

A wff will be true for an interpretation if all objects in the domain of quantification satisfy the predicates indicated in the wff. A wff will be logically true if it is true for all interpretations.

Let's take, for an example, the interpretation of a small set of sentences, with a small domain.

Sentences: 1. Pa • Pb
2. Wa • ~Wb
3. (∃x)Px
4. (∀x)Px
5. (∀x)(Wx ⊃ Px)
6. (∀x)(Px ⊃ Wx)

Domain: {Bob Simon, Rick Werner, Katheryn Doran, Todd Franklin, Marianne Janack, Russell Marcus, Martin Shuster}

a: Katheryn Doran
b: Bob Simon

Px: {Bob Simon, Rick Werner, Katheryn Doran, Todd Franklin, Marianne Janack, Russell Marcus, Martin Shuster}
Wx: {Katheryn Doran, Marianne Janack}

We can think of 'Px' as meaning that x is a professor of philosophy at Hamilton College. We can think of 'Wx' as meaning that x is a woman professor of philosophy at Hamilton College. But, the interpretation, speaking strictly, is made only extensionally, by the members of the sets listed.

We call an interpretation on which all of a set of given statements come out true a *model*. Given our interpretations of the predicates, not every sentence in our set is satisfied. 1-5 are satisfied. But, 6 is not. If we were to delete sentence 6 from our list, our interpretation would be a model.

To construct a model for a given set of sentences, we specify an interpretation, using the four steps above. (Only the first three require any thought, here, since we will assume the standard truth-tables for the connectives.)

Logical Truth and Validity

A wff will be *logically true* if it is true for every interpretation. For **PL**, the notion of logical truth was much simpler. All we had to do was look at the truth tables. For **M**, and even more so for **F**, the notion of logical truth is just naturally complicated by the fact that we are analyzing parts of propositions. Here are two logical truths of **M**.

LT1 (∀x)(Px ∨ ~Px)
LT2 Pa ∨ [(∀x)Px ⊃ Qa]

We can prove that LT1 and LT2 are logical truths by using our proof theory, or by using model-theoretic reasoning. Let's do LT1 by indirect proof.

│1. ~(∀x)(Px ∨ ~Px)	AIP
│2. (∃x)~(Px ∨ ~Px)	1, QE
│3. ~(Pa ∨ ~Pa)	2, EI
│4. ~Pa • ~~Pa	3, DM
5. (∀x)(Px ∨ ~Px)	1-4, IP, DN

QED

We can do LT2 by using a model-theoretic argument in the metalanguage.

> Suppose that 'Pa ∨ [(∀x)Px ⊃ Qa]' is not a logical truth.
> Then there is an interpretation on which it is false.
> On that interpretation, the object assigned to 'a' will not be in the set assigned to 'Px', and there is some counterexample to '(∀x)Px ⊃ Qa'
> Any counter-example to a conditional statement has to have a true antecedent.
> So, every object in the domain of our supposed interpretation will have to be in the set assigned to 'Px'.
> That contradicts the claim that the object assigned to 'a' will not be in the set assigned to 'Px'.
> So, our assumption must be false: no interpretation will make that sentence false.
> So, 'Pa ∨ [(∀x)Px ⊃ Qa]' is logically true.

QED

In the next section, I will discuss invalid arguments in predicate logic. A valid argument is one which is valid under any interpretation, using any domain. Our proof system has given us ways to show that an argument is valid. But when we introduced our system of inference for **PL**, we already had a way of distinguishing the valid from the invalid arguments, using truth tables. In **M**, we need a corresponding method for showing that an argument is invalid. An invalid argument will have counterexamples, interpretations on which the premises come out true and the conclusion comes out false. Understanding how we interpret theories in the language of predicate language will help us formulate a method for showing that an argument in predicate logic is invalid.

Exercises 3.6. Construct models for each of the following theories by specifying a domain of interpretation (make one up) and interpreting the constants and predicates.

1. Oa • ~Ob
   Ra • ~Ea
   Rd • Od • ~Ed
   (∃x)(Rx • Ox)
   ~(∃x)(Ex • Ox)
   (∃x)(Ex • Rx) ⊃ ~Oc

2. Mb • ~Md
   ~La • ~Wa
   Wc • Wd
   (∃x)(Mx • Lx)
   (∃x)(Mx • ~Wx)
   (∀x)(Lx ⊃ ~Wx)

3. Eb • Ec
   Kd • ~Ka
   ~Ea • Pa
   (∀x)(Ex ⊃ ~Kx)
   (∃x)(Px • Kx)
   (Eb ∨ Ed) ⊃ ~Ka

§3.7: Invalidity in Predicate Logic

We have studied a proof-theoretic method for showing that an argument in **M** is valid. We also need a method for showing that an argument in **M** is invalid.

Recall how we proved that an argument such as 3.7.1 is invalid in propositional logic.

3.7.1 1. A ⊃ B
2. ~(B • A) /A ≡ B

We lined up the premises and conclusion and assigned truth values to the component sentences to form a counterexample. A counterexample is a valuation which makes the premises true and the conclusion false.

3.7.2

A	⊃	B	/	~	(B	•	A)	//	A	≡	B
0	**1**	1		**1**	1	0	0		0	**0**	1

The table at 3.7.2 shows that the argument is invalid since there is a counterexample when A is false and B is true. We will adapt this method for first-order logic.

If an argument is valid, then it is valid no matter what we choose as our domain of interpretation. Logical truths are true for all interpretations. Even if our domain has only one member, or two or three, valid arguments have no counterexamples. Similarly, if an argument is invalid, then there will be a counterexample in some finite domain.

Universes of One Member

3.7.3 (∀x)(Wx ⊃ Hx)
(∀x)(Ex ⊃ Hx) / (∀x)(Wx ⊃ Ex)

Let's show that argument 3.7.3 is invalid. We will start by choosing a domain of one object in the universe. We will call it 'a'. Since there is only one object in the domain, the universally-quantified formulas are equivalent to statements about a.

3.7.4	(∀x)(Wx ⊃ Hx)	is equivalent to	Wa ⊃ Ha
	(∀x)(Ex ⊃ Hx)	is equivalent to	Ea ⊃ Ha
	(∀x)(Wx ⊃ Ex)	is equivalent to	Wa ⊃ Ea

We can thus eliminate the quantifiers and use the same method we used for arguments in **PL**. We assign truth values to make the premises true and the conclusion false, as in 3.7.5.

3.7.5

Wa	⊃	Ha	/	Ea	⊃	Ha	//	Wa	⊃	Ea
1	**1**	1		0	**1**	1		1	**0**	0

The argument 3.7.3 is shown invalid in a one-member universe, where Wa is true, Ha is true, and Ea is false. Again, a specification of the assignments of truth values to the atomic sentences of the theory, as in the previous sentence, is called a counterexample.

The method of finite universes works with existential quantifiers as well, as in argument 3.7.6.

3.7.6 1. $(\forall x)[Ux \supset (Tx \supset Wx)]$
2. $(\forall x)[Tx \supset (Ux \supset {\sim}Wx)]$
3. $(\exists x)(Ux \bullet Wx)$ $\quad / \ (\exists x)(Ux \bullet Tx)$

Expanding an existentially-quantified formula to a one-member universe, as in 3.7.7, works exactly like it does for universally-quantified formulas. In a world with just one thing, ‘everything’ is the same as ‘something’.

3.7.7			
	$(\forall x)[Ux \supset (Tx \supset Wx)]$	is equivalent to	$Ua \supset (Ta \supset Wa)$
	$(\forall x)[Tx \supset (Ux \supset {\sim}Wx)]$	is equivalent to	$Ta \supset (Ua \supset {\sim}Wa)$
	$(\exists x)(Ux \bullet Wx)$	is equivalent to	$Ua \bullet Wa$
	$(\exists x)(Ux \bullet Tx)$	is equivalent to	$Ua \bullet Ta$

The construction of a counterexample proceeds in the same way, too. The table at 3.7.8 shows that there is a counterexample in a one-member universe, where Ua is true; Ta is false; and Wa is true.

3.7.8

Ua	⊃	(Ta	⊃	Wa)	/	Ta	⊃	(Ua	⊃	~	Wa)	/	Ua	•	Wa	//	Ua	•	Ta
1	**1**	0	1	1		0	**1**	1	0	0	1		1	**1**	1		1	**0**	0

Be careful not to confuse expansions into finite universes with instantiation in natural deductions. In each case, we remove quantifiers. But, the restrictions on EI play no role in expansions.

All we need is one universe in with a counterexample to show that an argument is invalid. But not all invalid arguments are shown invalid in a one-member universe. Even if an argument has no counterexample in a one-member universe, it might still be invalid.

Universes of More Than One Member

Argument 3.7.9 has no counterexample in a one-member universe.

3.7.9 $(\forall x)(Wx \supset Hx)$
$(\exists x)(Ex \bullet Hx)$ $\quad / \ (\forall x)(Wx \supset Ex)$

Wa	⊃	Ha	/	Ea	•	Ha	//	Wa	⊃	Ea
				0				1	**0**	0

To make the conclusion false, we have to make ‘Wa’ true and ‘Ea’ false. Then, the second premise is false no matter what value we assign to ‘Ha’.

To show that 3.7.9 is invalid, we have to consider a larger universe. If there are two objects in a universe, a and b, then the expansions of quantified formulas become more complex. Universally quantified formulas become conjunctions because every object in the domain has whatever property the formula ascribes. Existentially quantified formulas become disjunctions because only some objects have the property ascribed by the formula. 3.7.10 shows the rules for expanding quantified formulas into two- and three-member domains.

3.7.10 In a two-member domain:

$(\forall x)\mathscr{F}x$ becomes $\mathscr{F}a \bullet \mathscr{F}b$
$(\exists x)\mathscr{F}x$ becomes $\mathscr{F}a \vee \mathscr{F}b$

In a three-member domain

$(\forall x)\mathscr{F}x$ becomes $\mathscr{F}a \bullet \mathscr{F}b \bullet \mathscr{F}c$
$(\exists x)\mathscr{F}x$ becomes $\mathscr{F}a \vee \mathscr{F}b \vee \mathscr{F}c$

Returning to argument 3.7.9, we will expand the argument into in a universe of two members and then look for a counterexample.

3.7.11 $(Wa \supset Ha) \bullet (Wb \supset Hb)$
$(Ea \bullet Ha) \vee (Eb \bullet Hb)$ $\quad / \; (Wa \supset Ea) \bullet (Wb \supset Eb)$

Assign values to each of the terms to construct a counterexample.

(Wa	⊃	Ha)	•	(Wb	⊃	Hb)
1	1	1	**1**	0	1	1

(Ea	•	Ha)	∨	(Eb	•	Hb)
0	0	1	**1**	1	1	1

//	(Wa	⊃	Ea)	•	(Wb	⊃	Eb)
	1	0	0	**0**	0	1	1

The counterexample for argument 3.7.9 can be read directly from the table at 3.7.11 and is summarized at 3.7.12.

3.7.12 There is a counterexample in a two-member universe, when:

Wa: true Wb: false
Ha: true Hb: true
Ea: false Eb: true

Constants

When expanding formulas into finite universes, constants get rendered as themselves. That is, we don't expand a term with a constant when moving to a larger universe. If an argument contains more than one constant, then it will require a domain larger than one object.

Remember that expanding formulas into finite universes is not the same as instantiating. In particular, the restriction on EI that we must instantiate to a new constant does not apply. If an argument contains both an existential quantifier and a constant, you may expand the quantifier into a single-member universe using the constant already present in the argument. It need not be a new constant.

3.7.13 can not be shown invalid in a one-member universe, despite having only one constant.

3.7.13 $\quad (\exists x)(Ax \bullet Bx)$
$\quad Ac \quad /Bc$

Ac	•	Bc	/	Ac	//	Bc
	0	0				0

We can generate a counterexample in a two-member universe, though, as at 3.7.14.

3.7.14

(Ac	•	Bc)	∨	(Aa	•	Ba)	/	**Ac**	//	**Bc**
1	0	0	**1**	1	1	1		**1**		**0**

There is a counterexample in a two-member universe, when:

Aa: true $\quad$ Ac: true
Ba: true $\quad$ Bc: false

Some arguments require three-member, four-member, or larger universes to be shown invalid.

Propositions Whose Main Operator is Not a Quantifier

The main operator of the second premise of 3.7.15 is a ⊃, not a quantifier. On each side of the conditional, there is a quantifier. There is no counterexample to the argument in a one-member universe, though the expansion is straightforward.

3.7.15 $\quad (\exists x)(Px \bullet Qx)$
$\quad (\forall x)Px \supset (\exists x)Rx$
$\quad (\forall x)(Rx \supset Qx) \quad / (\forall x)Qx$

Pa	•	Qa	/	Pa	⊃	Ra	/	Ra	⊃	Qa	//	**Qa**
	0	0										**0**

In a two-member universe, each quantifier in the second premise is unpacked independently, as in 3.7.16. Notice that the main operator of the premise remains the conditional.

3.7.16 $\quad (\forall x)Px \supset (\exists x)Rx \quad$ becomes $\quad (Pa \bullet Pb) \supset (Ra \vee Rb)$

We can clearly see here the difference between instantiation and expansion into a finite universe. We can not instantiate the formula, but we expand each quantifier.

We can construct a counterexample for the argument 3.7.15 in a two-member universe.

3.7.16

(Pa	•	Qa)	∨	(Pb	•	Qb)
	0	0	**1**	1	1	1

(Pa	•	Pb)	⊃	(Ra	∨	Rb)
		1	**1**	0	1	1

(Ra	⊃	Qa)	•	(Rb	⊃	Qb)
0	1	0	**1**	1	1	1

//	Qa	•	Qb
	0	0	1

There is a counterexample in a two-member universe, when:

Pa: either true or false	Pb: true
Qa: false	Qb: true
Ra: false	Rb: true

There is another counterexample which I leave as an exercise.

Exercises 3.7. Show each of the following arguments invalid by generating a counterexample.

1.
1. $(\exists x)(Ax \vee Bx)$
2. $(\forall x)Ax$ / $(\forall x)Bx$

2.
1. $(\forall x)(Cx \supset Dx)$
2. Da / Ca

3.
1. $(\exists x)(Ex \bullet Fx)$
2. Fb / Eb

4.
1. $(\forall x)(Kx \equiv Lx)$
2. $(\exists x)(Mx \bullet Lx)$ / $(\exists x)(Nx \bullet Kx)$

5.
1. $(\forall x)[(Gx \bullet Hx) \vee Ix]$
2. $({\sim}Hc \supset Jc) \supset {\sim}Ic$ / $(\exists x)(Gx \bullet {\sim}Jx)$

6.
1. $(\exists x)Dx \supset (\exists x)Gx$
2. $(\exists x)(Dx \bullet Ex)$ / $(\exists x)(Ex \bullet Gx)$

7.
1. $(\forall x)(Px \equiv Rx)$
2. $(\exists x)(Qx \bullet {\sim}Sx)$ / $(\forall x)(Qx \supset {\sim}Rx)$

8.
1. $(La \bullet {\sim}Lb) \bullet ({\sim}Mc \bullet Md)$
2. $(\exists x)(Lx \bullet Nx)$
3. $(\exists x)(Mx \bullet Ox)$
4. $(\forall x)[(Lx \vee Mx) \supset Ox]$ / $(\forall x)(Nx \supset Ox)$

9.
1. $(\exists x)[(Ax \bullet Bx) \bullet Cx]$
2. $(\exists x)[(Ax \bullet Bx) \bullet {\sim}Cx]$
3. $(\exists x)(Bx \bullet Dx)$
4. ${\sim}Da$ / $(\forall x)(Cx \supset Dx)$

10.
1. $Pa \bullet Qb$
2. $(\exists x)(Rx \bullet Sx)$
3. $(\exists x)(Rx \bullet {\sim}Sx)$
4. $(\forall x)(Sx \supset Qx)$ / $(\forall x)(Rx \supset Px)$

11.
1. $(\exists x)(Lx \bullet Nx)$
2. $(\exists x)(Mx \bullet {\sim}Nx)$
3. $(\forall x)(Lx \supset Ox)$ / $(\forall x)(Mx \supset Ox)$

12.
1. $(\exists x)(Ix \bullet Jx)$
2. $(\exists x)({\sim}Ix \bullet Jx)$
3. $(\forall x)(Jx \supset Kx)$ / $(\forall x)(Ix \supset Kx)$

13.
1. $(\exists x)(Ax \bullet Bx)$
2. $(\exists x)(Cx \bullet {\sim}Bx)$
3. $(\forall x)[(Ax \bullet Cx) \supset Dx]$ / $(\forall x)(Bx \supset Dx)$

14. 1. (∃x)(Rx ∨ ~Tx)
2. (∃x)(~Rx • Tx)
3. (∀x)(Sx ≡ Tx) / (∀x)(Sx ⊃ Rx)

15. 1. (∀x)Ax ⊃ (∀x)Bx
2. (∃x)(Ax • ~Bx)
3. (∀x)(Cx ⊃ Bx) / (∀x)(Cx ⊃ Ax)

16. 1. (∃x)[Ox • (Px ≡ Qx)]
2. (∃x)[~Ox • (Px ⊃ Qx)]
3. (∀x)(Rx ⊃ Ox) / (∀x)(Rx ⊃ Qx)

17. 1. (∃x)(Ex • Fx)
2. (∃x)(Ex • ~Fx)
3. (∀x)(Fx ≡ Gx) / (∀x)Ex

18. 1. (∀x)(Ax ⊃ Bx) ⊃ (∃x)Cx
2. (∃x)(Ax • ~Bx)
3. (∀x)(Dx ⊃ Bx) / (∀x)(Dx ⊃ Cx)

19. 1. (∃x)Ex ⊃ (∃x)Fx
2. (∃x)(Ex • ~Fx)
3. (∀x)[(Gx ∨ Hx) ⊃ Fx] / (∀x)(Hx ⊃ Ex)

20. 1. (∀x)(Jx ≡ Ix) • (∃x)Kx
2. (∃x)(Ix • ~Kx)
3. (∀x)(Lx ⊃ Kx)
4. ~Ja • Jb / (∀x)(Lx ⊃ Ix)

21. 1. (∀x)(Mx ⊃ Nx)
2. (∃x)(~Nx • Ox)
3. (∀x)(Px ⊃ ~Ox)
4. Ma • Mb / (∀x)(Px ⊃ Nx)

22. 1. (∃x)Tx ⊃ (∃x)Sx
2. (∃x)(~Sx • Tx)
3. (∀x)(Ux ⊃ Sx) / (∀x)(Ux ⊃ Tx)

23. 1. (∃x)(Nx • Ox)
2. (∃x)(~Nx • Px)
3. (∀x)(Px ⊃ Qx) / (∀x)(Nx ⊃ Qx)

24. 1. (∃x)(~Hx • Ix)
2. (∃x)(Hx • ~Ix)
3. (∀x)(Jx ≡ Ix) / (∀x)(Hx ⊃ Jx)

25. 1. (∃x)(Kx • Mx)
2. La • Lb / (∃x)(Lx ≡ Mx)

26. 1. $(\exists x)(\sim Ax \equiv Cx)$
2. $(\exists x)(Ax \bullet Cx)$
3. $(\forall x)(Bx \supset Ax)$ / $(\forall x)(Cx \supset Bx)$

27. 1. $(Ha \bullet \sim Ia) \bullet Ja$
2. $(\exists x)[Ix \bullet (Jx \equiv \sim Kx)]$
3. $(\exists x)(\sim Jx \vee Kx)$ / $(\exists x)Kx$

28. 1. $(\forall x)(Px \supset Qx)$
2. $(\forall x)(\sim Px \equiv Rx)$
3. $(\exists x)(Qx \bullet Rx)$
4. $(Pa \bullet Pb) \bullet (\sim Pc \bullet \sim Pd)$ / $\sim Qb$

29. 1. $(\exists x)(Hx \bullet Ix)$
2. $(\exists x)(Hx \bullet \sim Ix)$
3. $(\forall x)(Jx \supset Ix)$ / $(\forall x)(Jx \supset Hx)$

30. 1. $(\exists x)(Kx \bullet \sim Lx)$
2. $(\exists x)(Kx \bullet Lx)$
3. $(\forall x)[(Mx \vee Nx) \supset Lx]$ / $(\forall x)(Mx \supset Kx)$

31. 1. $(\exists x)Sx$
2. $(\forall x)[Sx \supset (Tx \supset \sim Ux)]$
3. $Ua \bullet Ub$
4. $(\exists x)\sim Ux$ / $(\exists x)(Sx \bullet \sim Tx)$

32. 1. $(\exists x)(Ax \bullet Bx)$
2. $(\exists x)[(Ax \bullet \sim Bx) \bullet Cx]$
3. $(\forall x)(\sim Ax \supset Dx)$
4. $(\forall x)(Dx \supset Cx)$ / $Ca \vee Cb$

33. 1. $(\exists x)(Fx \bullet Gx)$
2. $(\exists x)(\sim Fx \bullet Gx)$
3. $(\forall x)[Gx \supset (Fx \equiv Hx)]$ / $(\forall x)(Fx \equiv Hx)$

34. 1. $(\exists x)[(Ex \bullet Fx) \bullet Gx]$
2. $(\exists x)[(Ex \bullet \sim Fx) \bullet Gx]$
3. $(\exists x)(\sim Ex \bullet Gx)$
4. $(\forall x)(Gx \supset Hx)$
5. $(\forall x)(\sim Gx \supset \sim Ex)$ / $Ha \vee Fa$

35. 1. $(\exists x)(Px \bullet Qx)$
2. $(\exists x)(Rx \equiv \sim Sx)$
3. $(\exists x)Sx$
4. $(\forall x)[(Qx \vee Rx) \supset \sim Sx]$ / $(\exists x)(Qx \bullet \sim Rx)$

§3.8: Translation Using Relational Predicates

Argument 3.1.1 showed that some intuitively-valid inferences were not valid in **PL**. We moved to **M** in response. Argument 3.8.1 shows that some intuitively-valid inferences are not valid in **M** and that we should look at a further refinement of our logic.

3.8.1
Andrew is taller than Bob.
Bob is taller than Charles.
For any x, y and z, if x is taller than y and y is taller than z, then x is taller than z.
So, Andrew is taller than Charles.

In **M**, with only monadic predicates, we translate the two first sentences with different predicates. The first sentence ascribes the property of being taller than Bob to Andrew. The second sentence ascribes the property of being taller than Charles to Bob. Being taller than Charles is of course a different property from being taller than Bob.

3.8.2	Andrew is taller than Bob	Ua
	Bob is taller than Charles	Tb

In examining the argument 3.8.1, we can see that what we really want is a more-general predicate, being taller than, that takes two objects. Such a predicate is called dyadic. 3.8.3 contains examples of various dyadic predicates.

3.8.3
Txy: x is taller than y
Kxy: x knows y
Bxy: x believes y
Dxy: x does y

We can construct three-place predicates too, called triadic predicates, as at 3.8.4.

3.8.4
Gxyz: x gives y to z
Kxyz: x kisses y in z
Bxyz: x is between y and z

Further, we can construct four-place and higher-place predicates. All predicates which take more than one object are called relational, or polyadic. With relational predicates, we now have a choice how to regiment relations.

3.8.5 Andrés loves Beatriz

We could regiment 3.8.5 in monadic predicate logic as ‘La’. In that case, ‘a’ stands for Andrés, and ‘L’ stands for the property of loving Beatriz. But, if we want to use ‘L’ to stand for the property of loving, it will have to take two objects: the lover and the lovee. We can introduce a relational predicate, ‘Lxy’, which means that x loves y. Then, we regiment 3.8.5 as 3.8.6.

3.8.6 Lab

A similar translation, using a three-place relation for giving, can help us avoid using an overly-simple monadic predicate for 3.8.7.

3.8.7 Camila gave David the earring.

Instead of using ‘Gx’ for ‘x gives David the earring’, we can invoke ‘Gxyz’ for ‘x gives y to z’. Then, 3.8.7 is regimented as 3.8.8.

3.8.8 Gced

By using relational predicates, we reveal more logical structure. The more logical structure we reveal, the more we can facilitate inferences. We will mostly, in this text, use two- and three-place relations. But, more-complex relations can be useful. For example, 3.8.9, couched in a serious scientific theory, might be regimented using a five-place relation.

3.8.9 There is something blue over there now.

We need one place for the object. To indicate its spatial position, we could use three places: one for each position on a three-dimensional coordinate axis. We add one more place for a temporal location. 3.8.10 uses constants for spatial and temporal locations, but we could of course quantify over them as well.

3.8.10 (∃x)Bxabct

In other words, something is blue at spatial location a,b,c at time t. The utility of a language with more variables (and thus more quantifier variables) should be apparent.

By introducing relational predicates, we have extended our language. We are now using a language I call **F**, for Full First-Order Predicate logic, rather than **M**. The differences here between **F** and **M** are minor. The two languages use the same vocabulary. The only significant difference in the formation rules is in the construction of atomic formulas.

Beyond this text, the differences between **M** and **F** are significant; we have breached a barrier. **M** admits of a decision procedure. If a theory admits of a decision procedure, there is a way of deciding, for any given formula, whether it is a theorem or not. **F** is not decidable. There are formulas for which there are no effective methods for deciding whether they are theorems or not.

Formation rules for wffs of F

1. An n-place predicate followed by n singular terms is a wff.
2. For any variable β, if α is a wff that does not contain either ‘(∃β)’ or ‘(∀β)’, then ‘(∃β)α’ and ‘(∀β)α’ are wffs.
3. If α is a wff, so is ~α.
4. If α and β are wffs, then so are:
   (α • β)
   (α ∨ β)
   (α ⊃ β)
   (α ≡ β)
5. These are the only ways to make wffs.

Remember that singular terms, for now, are either constants or variables; later we will add functions.

You can determine the value of 'n' in an n-place predicate just by counting the number of singular terms that follow the predicate letter. In 3.8.11, there are one-place, two-place, three-place, and four-place predicates. Despite using the same predicate-letter, 'P', these predicates are all different.

3.8.11 Pa
Pab
Pabc
Pabcd

The semantics of **M** must also be adjusted to account for relational predicates. Recall that there were four steps for providing a standard formal semantics for **M**.

Step 1. Specify a set to serve as a domain of interpretation, or domain of quantification.
Step 2. Assign a member of the domain to each constant.
Step 3. Assign some set of objects in the domain to each predicate.
Step 4. Use the customary truth tables for the interpretation of the connectives.

The introduction of relational predicates requires adjustment to Step 3. For an interpretation of **F**, we assign sets of ordered n-tuples to each relational predicate.

An n-tuple is an n-place relation. Essentially, it's an ordered sequence of objects, a set with structure. The sets {1, 2} and {2, 1} are equivalent, since all that matters for the constitution of a set is its members. In contrast, the ordered triple <1, 2, 5> is distinct from the ordered triple <2, 1, 5> which is distinct from the ordered triple <5, 2, 1> even though they all have the same members.

For the semantics of **F**, a two-place predicate is assigned sets of ordered pairs, a three-place predicate is assigned sets of three-place relations, etc. Given a domain of {1, 2, 3}, the relation 'Gxy', which could be understood as meaning 'is greater than' would be standardly interpreted by {<2,1>, <3,1>, <3, 2>}.

Our definitions of satisfaction and truth will need to be adjusted. Objects in the domain can satisfy predicates; that remains the case for one-place predicates. Ordered n-tuples may satisfy relational predicates. A wff will be satisfiable if there are objects in the domain of quantification which stand in the relations indicated in the wff. A wff will be true for an interpretation if all objects in the domain of quantification stand in the relations indicated in the wff. And, still, a wff will be logically true if it is true for every interpretation.

For an example, let's extend the interpretation of a small set of sentences with a small domain that we considered when originally discussing semantics of **M**, in §3.6.

Sentences:
1. Pa • Pb
2. Wa • ~Wb
3. Oab
4. Obc
5. (∃x)(Px • Oxb)
6. (∃x)(Px • Obx)
7. (∀x)[Wx ⊃ (∃y)(Px • Oyx)]

Domain: {Bob, Rick, Katheryn, Todd, Marianne, Russell, Martin}

a:	Katheryn
b:	Bob
c:	Russell
Px:	{Bob, Rick, Katheryn, Todd, Marianne, Russell, Martin}
Wx:	{Katheryn, Marianne}
Oxy:	{<Bob, Rick>, <Bob, Katheryn>, <Bob, Todd>, <Bob, Marianne>, <Bob, Russell>, <Bob, Martin>, <Rick, Katheryn>, <Rick, Todd>, <Rick, Marianne>, <Rick, Russell>, <Rick, Martin>, <Katheryn, Todd>, <Katheryn, Marianne>, <Katheryn, Russell>, <Katheryn, Martin>, <Todd, Marianne>, <Todd, Russell>, <Todd, Martin>, <Marianne, Russell>,<Marianne, Martin>, <Russell, Martin>}

On this interpretation, 1 and 2 are true; 3 is false while 4 is true; 5 is false but 6 and 7 are true.

Quantifiers with relational predicates

We can now translate the first two premises of 3.8.1 and the conclusion.

3.8.12	Andrew is taller than Bob.	Tab
	Bob is taller than Charles.	Tbc
	Andrew is taller than Charles.	Tac

To regiment the third premise of 3.8.1, we need multiple, overlapping quantifiers. Let's see how to use quantifiers with relational predicates in steps. We'll start with sentences with just one quantifier. The sentences at 3.8.13 use 'Bxy' for 'x is bigger than y'.

3.8.13	Joe is bigger than some thing.	(∃x)Bjx
	Something is bigger than Joe.	(∃x)Bxj
	Joe is bigger than everything.	(∀x)Bjx
	Everything is bigger than Joe.	(∀x)Bxj

Next, we can introduce overlapping quantifiers. 3.8.14 uses 'Lxy' for 'x loves y'.

3.8.14	Everything loves something.	(∀x)(∃y)Lxy

Note the different quantifier letters: overlapping quantifiers must use different variables. Also, the order of quantifiers matters. 3.8.15 differs from 3.8.14.

3.8.15 Something loves everything. $(\exists x)(\forall y)Lxy$

Switching the order of the quantifiers in front of a formula changes its meaning. We now can regiment 3.8.1 completely, as the argument 3.8.16.

3.8.16
1. Tbc
2. Tab
3. $(\forall x)(\forall y)(\forall z)[(Txy \bullet Tyz) \supset Txz]$ / Tac

We will return to deriving the conclusion of this argument in §3.10. For the remainder of this section, and in the next section as well, we will look at some more complicated translations.

3.8.17	Something teaches Plato. (Txy: x teaches y)	$(\exists x)Txp$
3.8.18	Someone teaches Plato. (Px: x is a person)	$(\exists x)(Px \bullet Txp)$
3.8.19	Plato teaches everyone.	$(\forall x)(Px \supset Tpx)$
3.8.20	Everyone teaches something.	$(\forall x)[Px \supset (\exists y)Txy]$
3.8.21	Some people teach themselves.	$(\exists x)(Px \bullet Txx)$

Let's take 'teacher' to refer to someone who teaches something and 'student' to refer to someone who is taught (by something). Then, we can use 'Txy', for 'x teaches y' to characterize both teachers and students.

3.8.22	There are teachers.	$(\exists x)(\exists y)Txy$
3.8.23	There are students.	$(\exists x)(\exists y)Tyx$
3.8.24	Skilled teachers are interesting.	$(\forall x)[(\exists y)Txy \supset (Sx \supset Ix)]$
3.8.25	Skilled teachers are better than unskilled teachers.	
	$(\forall x)\{[(\exists y)Txy \bullet Sx] \supset \{(\forall z)[(\exists w)Tzw \bullet \sim Sz] \supset Bxz\}\}$	

When you have multiple quantifiers in a proposition, they can take wide scope by standing in front of the proposition, as in 3.8.26. Or they can take narrow scope by being located inside the proposition, as in 3.8.27.

3.8.26 $(\exists x)(\exists y)[(Px \bullet Py) \bullet Lxy]$
3.8.27 $(\exists x)[Px \bullet (\exists y)(Py \bullet Lxy)]$

When translating, it is best form only to introduce quantifiers when needed. Give your quantifiers as narrow a scope as possible. On occasion, we will put all quantifiers in front of a formula, using wide scope. But, moving quantifiers around is not always simple, and we must be careful. 3.8.26 and 3.8.27 are logically equivalent. But, 3.8.28 and 3.8.29 are not.

3.8.28 $(\forall x)[Px \supset (\exists y)(Py \bullet Qxy)]$
3.8.29 $(\exists y)(\forall x)[Px \supset (Py \bullet Qxy)]$

3.8.28 could be used to regiment 'all people love someone'. Using the same interpretation of the predicates, 3.8.29 would stand for 'there is someone everyone loves'. 3.8.28 is plausible. 3.8.29 is not. There are rules for moving quantifiers through a formula without altering the meaning and which we will discuss in the next section.

Exercises 3.8a. Translate each sentence into predicate logic using relational predicates.

1. David teaches Chris. (Txy: x teaches y)

2. Earth isn't bigger than Jupiter. (Bxy: x is bigger than y)

3. Frances loves Holly. (Lxy: x loves y)

4. Leo is taller than Cathy. (Txy: x is taller than y)

5. Alexis is greeted by Ben. (Gxy: x greets y)

6. George borrows Hector's lawnmower. (Bxyx: x borrows y from z)

7. José introduces Wilma to Kevin. (Ixyz: x introduces y to z)

8. Marco moves from Italy to Spain. (Mxyz: x moves to y from z)

9. Nicole does not drive from her office to her school. (Dxyz: x drives from y to z)

10. Ricardo gives the wedding ring to Olivia. (Gxyz: x gives y to z)

Exercises 3.8b. Translate each of the following into predicate logic using relational predicates.

1. Someone is smarter than Albert. (a, Px, Sxy: x is smarter than y)

2. Everyone is smarter than Albert.

3. Albert is smarter than everyone.

4. No one is smarter than Albert.

5. Everyone is smarter than someone.

6. Someone is smarter than everyone.

7. All rats are larger than Ben. (b, Rx, Lxy: x is larger than y)

8. Ben is larger than all rats.

9. Some rats are larger than Ben.

10. No rats are larger than Ben.

11. All rats are larger than some mice. (Mx, Rx, Lxy: x is larger than y)

12. Some rats are larger than some hamsters. (Hx, Rx, Lxy: x is larger than y)

13. Orsola introduces Chiara to Marina. (c, m, o, Ixyz: x introduces y to z)

14. Someone introduces Chiara to Marina. (c, m, Px, Ixyz: x introduces y to z)

15. Someone introduces Chiara to everyone. (c, Px, Ixyz: x introduces y to z)

16. All planets are smaller than Jupiter. (j, Px, Sxy: x is smaller than y)

17. All runners are healthier than some people. (Px, Rx, Hxy: x is healthier than y)

18. All kings are luckier than all paupers. (Kx, Px, Lxy: x is luckier than y)

19. All birds live in some nest. (Bx, Nx, Lxy: x lives in y)

20. Some CEO is wealthier than everyone. (Cx, Px, Wxy: x is wealthier than y)

21. Some cheetahs are faster than all lions. (Cx, Lx, Fxy: x is faster than y)

22. No tiger is faster than all cheetahs. (Cx, Tx, Fxy: x is faster than y)

23. No lion is faster than some tigers. (Lx, Tx, Fxy: x is faster than y)

24. Some dancers are thinner than some people. (Dx, Px, Txy: x is thinner than y)

25. All dancers are fitter than some people. (Dx, Px, Fxy: x is fitter than y)

26. No dancer is clumsier than some football player. (Dx, Fx, Cxy: x is clumsier than y)

27. All jellybeans are sweeter than all fruit. (Fx, Jx, Sxy: x is sweeter than y)

28. All children deserve to go to school. (Cx, Sx, Dxy: x deserves y)

29. All math majors take a calculus class. (Cx, Mx, Txy: x takes y)

30. Most math majors take a statistics class. (Mx, Sx, Txy: x takes y)

31. Some robbers steal money from a bank. (Bx, Mx, Rx, Sxyz: x steals y from z)

32. Some kind people help some endangered species. (Ex, Kx, Px, Sx, Hxy: x helps y)

33. Some cruel people starve animals. (Ax, Cx, Px, Sxy: x starves y)

34. Some firemen rescue victims from danger. (Fx, Vx, Dx, Rxyz: x rescues y from z)

35. Jen reads all books written by Asimov. (j, a, Bx, Rxy: x reads, Wxy: x writes y)

36. Some people read all books written by Asimov. (Bx, Px, Rxy: x reads, Wxy: x writes y)

37. No god is mightier than herself. (Gx, Mxy: x is mightier than y)

38. No sword is mightier than any pen. (Px, Sx, Mxy: x is mightier than y)

39. Everyone buys something from some merchant. (Mx, Px, Bxyz: x buys y from z)

40. No merchant has everyone for a customer.

Exercises 3.8c. Use the translation key to translate the formulas into natural English sentences.[2]

Ax: x is silver
Bxy: x belongs to y
Cx: x is a cloud
Cxy: x keeps company with y
Dx: x is a dog
Ex: x is smoke
Fx: x is fire
Fxy: x is fair for y
g: God
Gx: x is glass
Gxy: x gathers y
Hx: x is home
Hxy: x helps y
Ixy: x is in y
Jxy: x is judged by y
Kxy: x is a jack of y
Lx: x is a lining
Lxy: x is like y
Mx: x is moss
Mxy: x is master of y
Px: x is a person
Qx: x is a place
Rx: x rolls
Sx: x is a stone
Tx: x is a trade
Txy: x should throw y
Ux: x is a house
Uxy: x comes to y
Vxy: x ventures y
Wx: x waits
Yx: x is a day

1. $(\forall x)[Dx \supset (\exists y)(Yy \cdot Byx)]$

2. $(\forall x)[(\exists y)(Py \cdot Fxy) \supset (\forall z)(Pz \supset Fxz)]$

3. $(\forall x)[(Rx \cdot Sx) \supset (\forall y)(My \supset {\sim}Gxy)]$

4. $(\forall x)[(Px \cdot Wx) \supset (\forall y)Uyx]$

5. $(\forall x)[(Px \cdot Hxx) \supset Hgx]$

6. $(\forall x)[Hx \supset (\forall y)(Qy \supset {\sim}Lyx)]$

7. $(\forall x)\{Cx \supset (\exists y)[(Ay \cdot Ly) \cdot Byx]\}$

8. $(\forall x)[Px \supset (\forall y)(Cxy \supset Jxy)]$

9. $(\forall x)\{Qx \supset [(\exists y)(Ey \cdot Iyx) \supset (\exists z)(Fz \cdot Izx)]\}$

10. $(\forall x)\{[Px \cdot (\forall y)(Ty \supset Kxy)] \supset (\forall z)(Tz \supset {\sim}Mxz)\}$

11. $(\forall x)\{\{Px \cdot (\exists y)[(Gy \cdot Uy) \cdot Ixy]\} \supset (\forall z)(Sz \supset {\sim}Txz)\}$

12. $(\forall x)\{[Px \cdot (\forall y){\sim}Vxy] \supset (\forall z){\sim}Gxz\}$

[2] Adapted from Copi, *Symbolic Logic*, 5th ed., MacMillan Publ., 1979, pp 127-8.

§3.9: Rules of Passage

Quantifiers: narrow and wide scope

I mentioned in the last section that when translating, it is good form give your quantifiers narrow scope. Further, moving quantifiers through a formula can alter the meaning of the formula. In this section, we will look at rules for moving quantifiers through formulas and practice some translation in the process.

In some cases, we can move quantifiers around without much worry. For example, if all quantifiers are universal, we can pull them in or out at will, as long as we are careful not to accidentally bind any variables. 3.9.1 can be written as any of 3.9.2 - 3.9.4.

3.9.1	Everyone loves everyone
3.9.2	$(\forall x)[Px \supset (\forall y)(Py \supset Lxy)]$
3.9.3	$(\forall x)(\forall y)[(Px \bullet Py) \supset Lxy]$
3.9.4	$(\forall y)(\forall x)[(Px \bullet Py) \supset Lxy]$

Technically, 3.9.4 is 'everyone is loved by everyone'. But all three statements are logically equivalent. Similarly, 3.9.5 can be written as any of 3.9.6 - 3.9.8.

3.9.5	Someone loves someone.
3.9.6	$(\exists x)[Px \bullet (\exists y)(Py \bullet Lxy)]$
3.9.7	$(\exists x)(\exists y)[(Px \bullet Py) \bullet Lxy]$
3.9.8	$(\exists y)(\exists x)[(Px \bullet Py) \bullet Lxy]$

3.9.8 is 'someone is loved by someone'. But, again, 3.9.6 - 3.9.8 are all logically equivalent.

When you mix universal quantifiers with existential quantifiers, you must be careful. Slight changes, like reversing the order of the quantifiers, can change the meaning of a proposition. None of 3.9.9 - 3.9.12 are equivalent.

3.9.9	Everyone loves someone.	$(\forall x)(\exists y)[Px \supset (Py \bullet Lxy)]$
3.9.10	Everyone is loved by someone.	$(\forall x)(\exists y)[Px \supset (Py \bullet Lyx)]$
3.9.11	Someone loves everyone.	$(\exists x)(\forall y)[Px \bullet (Py \supset Lxy)]$
3.9.12	Someone is loved by everyone.	$(\exists x)(\forall y)[Px \bullet (Py \supset Lyx)]$

Note that the first word in each translation above corresponds to the leading quantifier. Also, note that the connectives which directly follow the 'Px' and the 'Py' are determined by the quantifier binding that variable. This tendency is clearer if we take the quantifiers inside.

3.9.9'	$(\forall x)[Px \supset (\exists y)(Py \bullet Lxy)]$
3.9.10'	$(\forall x)[Px \supset (\exists y)(Py \bullet Lyx)]$
3.9.11'	$(\exists x)[Px \bullet (\forall y)(Py \supset Lxy)]$
3.9.12'	$(\exists x)[Px \bullet (\forall y)(Py \supset Lyx)]$

While all of these shifts of quantifiers are acceptable, moving quantifiers within a proposition is tricky. For example, 3.9.13 and 3.9.14 are *not* equivalent, as a possible interpretation of each shows.

3.9.13	$(\forall x)[(\exists y)Lxy \supset Hx]$	All lovers are happy
3.9.14	$(\forall x)(\exists y)(Lxy \supset Hx)$	Everything has something such that loving it will make it (the lover) happy.

From 3.9.13 to 3.9.14, we have moved the existential quantifier out front, and merely brought the 'Hx' into the scope of '$(\exists y)$', which does not bind it. 3.9.13 does not commit to the existence of something that, by being loved, makes something happy. 3.9.14 does. Consider the universe in which there are things that can never be happy, i.e. for which nothing could make them happy. 3.9.13 could still be true, but 3.9.14 would be false.

We need a set of rules to determine which moves of quantifiers are acceptable. Also motivating the need for such rules, there are metalogical proofs which require that every statement of **F** is equivalent to a statement with all quantifiers having wide scope. Such a form is called prenex normal form (PNF). In order to transform formulas to PNF, we can use what are sometimes called rules of passage. The rules of passage are rules of equivalence, justified by the equivalence of statements of the paired forms and applicable to whole lines or to parts of lines.[3]

Rules of Passage

For all variables α and all formulas Γ and Δ:

RP1: $(\exists\alpha)(\Gamma \vee \Delta) \rightleftarrows (\exists\alpha)\Gamma \vee (\exists\alpha)\Delta$
RP2: $(\forall\alpha)(\Gamma \bullet \Delta) \rightleftarrows (\forall\alpha)\Gamma \bullet (\forall\alpha)\Delta$

For all variables α, all formulas Γ containing α, and all formulas Δ not containing α:

RP3: $(\exists\alpha)(\Delta \bullet \Gamma\alpha) \rightleftarrows \Delta \bullet (\exists\alpha)\Gamma\alpha$
RP4: $(\forall\alpha)(\Delta \bullet \Gamma\alpha) \rightleftarrows \Delta \bullet (\forall\alpha)\Gamma\alpha$

RP5: $(\exists\alpha)(\Delta \vee \Gamma\alpha) \rightleftarrows \Delta \vee (\exists\alpha)\Gamma\alpha$
RP6: $(\forall\alpha)(\Delta \vee \Gamma\alpha) \rightleftarrows \Delta \vee (\forall\alpha)\Gamma\alpha$

RP7: $(\exists\alpha)(\Delta \supset \Gamma\alpha) \rightleftarrows \Delta \supset (\exists\alpha)\Gamma\alpha$
RP8: $(\forall\alpha)(\Delta \supset \Gamma\alpha) \rightleftarrows \Delta \supset (\forall\alpha)\Gamma\alpha$

RP9: $(\exists\alpha)(\Gamma\alpha \supset \Delta) \rightleftarrows (\forall\alpha)\Gamma\alpha \supset \Delta$
RP10: $(\forall\alpha)(\Gamma\alpha \supset \Delta) \rightleftarrows (\exists\alpha)\Gamma\alpha \supset \Delta$

When moving quantifiers using the rules of passage, be careful not to have any accidental binding, or accidental removing from binding.

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[3] Quine notes that the rules of passage were so-called by Herbrand, in 1930, but were present in Whitehead and Russell's *Principia Mathematica*. Prenex normal form was used by Skolem for his proof procedure, in 1922.

Let's look at a few examples. 3.9.15 and 3.9.16 are equivalent by RP4.

3.9.15	$(\exists x)[Px \bullet (\forall y)(Qy \supset Rxy)]$
3.9.16	$(\exists x)(\forall y)[Px \bullet (Qy \supset Rxy)]$

If we didn't have RP4, we could show their equivalence by deriving 3.9.15 from 3.9.16 and 3.9.16 from 3.9.15. '$\alpha \vdash \beta$' means that β can be derived from α; '$\vdash$' is the metalinguistic form of '$\supset$' I will write its negation as '$\sim\vdash$'. (We haven't discussed how the rules of inference have to be restricted when using relational predicates, but the change is small and all of the derivations in this section are acceptable.)

3.9.15 $\vdash$ 3.9.16

1. $(\exists x)[Px \bullet (\forall y)(Qy \supset Rxy)]$	
2. $Pa \bullet (\forall y)(Qy \supset Ray)$	1, EI
| 3. Qy	ACP
| 4. $(\forall y)(Qy \supset Ray)$	2, Com, Simp
| 5. $Qy \supset Ray$	4, UI
| 6. Ray	5, 3, MP
7. $Qy \supset Ray$	3-6, CP
8. Pa	2, Simp
9. $Pa \bullet (Qy \supset Ray)$	8, 7, Conj
10. $(\forall y)[Pa \bullet (Qy \supset Ray)]$	9, UG
11. $(\exists x)(\forall y)[Px \bullet (Qy \supset Rxy)]$	10, EG

QED

3.9.16 $\vdash$ 3.9.15

1. $(\exists x)(\forall y)[Px \bullet (Qy \supset Rxy)]$	
2. $(\forall y)[Pa \bullet (Qy \supset Ray)]$	1, EI
3. $Pa \bullet (Qy \supset Ray)$	2, UI
4. $Qy \supset Ray$	3, Com, Simp
5. $(\forall y)(Qy \supset Ray)$	4, UG
6. Pa	3, Simp
7. $Pa \bullet (\forall y)(Qy \supset Ray)$	6, 5, Conj
8. $(\exists x)[Px \bullet (\forall y)(Qy \supset Rxy)$	7, EG

QED

3.9.17 and 3.9.18 are equivalent by RP8:

3.9.17	$(\exists x)(\forall y)[Px \supset (Qy \supset Rxy)]$
3.9.18	$(\exists x)[Px \supset (\forall y)(Qy \supset Rxy)]$

3.9.14, above, is equivalent to 3.9.19 by RP9.

3.9.14	$(\forall x)(\exists y)(Lxy \supset Hx)$
3.9.19	$(\forall x)[(\forall y)Lxy \supset Hx]$

The transformation between 3.9.14 and 3.9.19 might strike one as strange. It might even make one call RP9 into question. But, notice that we can make the same transformation without RP9.

3.9.14	1. $(\forall x)(\exists y)(Lxy \supset Hx)$	
	2. $(\forall x)(\exists y)({\sim}Lxy \vee Hx)$	1, Impl
	3. $(\forall x)(\exists y)(Hx \vee {\sim}Lxy)$	2, Com
	4. $(\forall x)[Hx \vee (\exists y){\sim}Lxy]$	3, RP5
	5. $(\forall x)[(\exists y){\sim}Lxy \vee Hx]$	4, Com
	6. $(\forall x)[{\sim}(\forall y)Lxy \vee Hx]$	5, QE
3.9.19	7. $(\forall x)[(\forall y)Lxy \supset Hx]$	6, Impl

3.9.13, above, is equivalent by RP10 to 3.9.20.

3.9.13	$(\forall x)[(\exists y)Lxy \supset Hx]$
3.9.20	$(\forall x)(\forall y)(Lxy \supset Hx)$

Both formulas can be interpreted as, "If anyone loves someone, then s/he is happy."
3.9.21 and 3.9.22 are equivalent, also by RP10.

3.9.21	$(\forall x)[Px \supset (\exists y)Qy]$
3.9.22	$(\exists x)Px \supset (\exists y)Qy$

Proving the equivalence of RP10

We will not prove the equivalence of all of the Rules of Passage. Most of them are quite intuitive. RP9 and RP10 are the two oddballs. Let's take a moment to prove RP10.

$$\text{RP10: } (\forall\alpha)(\Gamma\alpha \supset \Delta) \qquad \rightleftarrows \qquad (\exists\alpha)\Gamma\alpha \supset \Delta$$

Consider first what happens when Δ is true, and then when Δ is false. (As an example, in 3.9.21, Δ is '$(\exists y)Qy$'.)

If Δ α is true, then both formulas will turn out to be true.
The consequent of the formula on the right is just Δ.
So, if Δ is true, the whole formula on the right is true.
$\Gamma\alpha \supset \Delta$ is true for every instance of α, since the consequent is true.
So, the universal generalization of each such formula (the formula on the left) is true.
If Δ is false, then the truth value of each formula will depend.
To show that the truth values of each formula will be the same, we will show that the formula on the right is true in every case that the formula on the left is true and that the formula on the left is true in every case that the formula on the right is.
If the formula on the left turns out to be true when Δ is false, it must be because $\Gamma\alpha$ is false, for every α.
But then, $(\exists\alpha)\Gamma\alpha$ is false, and so the formula on the right turns out to be true.
If the formula on the right turns out to be true, then it must be because $(\exists\alpha)\Gamma\alpha$ is false.
And so, there will be no value of α that makes $\Gamma\alpha$ true, and so the formula on the right will also turn out to be (vacuously) true.
QED

Using the Rules of Passage in Translations

RP10 allows us to translate 3.9.23 as 3.9.24 or as 3.9.25; the latter two are thus logically equivalent formulas.

3.9.23	If anything was damaged, then everyone gets upset	
3.9.24	$(\exists x)Dx \supset (\forall x)(Px \supset Ux)$	
3.9.25	$(\forall x)[Dx \supset (\forall y)(Py \supset Uy)]$	

Using the rules of passage, we can transform any statement of predicate logic into prenex normal form, with all the quantifiers out front. 3.9.26 uses only monadic predicates; still the rules of passage apply.

3.9.26	If there are any wildebeest, then if all lions are hungry, they will be eaten.	
3.9.27	$(\forall x)\{Wx \supset [(\forall y)(Ly \supset Hy) \supset Ex]\}$	
3.9.28	$(\forall x)\{Wx \supset (\exists y)[(Ly \supset Hy) \supset Ex]\}$	by RP9
3.9.29	$(\forall x)(\exists y)\{Wx \supset [(Ly \supset Hy) \supset Ex]\}$	by RP7

3.9.27 is the most natural translation of 3.9.26. It would be unlikely that any one would translate 3.9.26 as either 3.9.28 or 3.9.29. But our rules of inference only allow us to remove quantifiers from whole lines (i.e. when they are the main operators). So for the purposes of derivations, it may be useful to have the quantifiers out front.

Prenex Normal Form

It is not the case that a given formula has a unique prenex form. For example, 3.9.30 comes from Quine.

3.9.30 If there is a philosopher whom all philosophers contradict, then there is a philosopher who contradicts himself.

3.9.31 $(\exists x)[Fx \bullet (\forall y)(Fy \supset Gyx)] \supset (\exists x)(Fx \bullet Gxx)$

In order to put this sentence into prenex form, we have first to change the latter 'x's to 'z's, so that when we stack the quantifiers in front, we won't get accidental binding.

3.9.32 $(\exists x)[Fx \bullet (\forall y)(Fy \supset Gyx)] \supset (\exists z)(Fz \bullet Gzz)$

In the first set of transformations to prenex form, I will work with the 'z', then the 'y'.

	$(\exists z)(\exists x)\{[Fx \bullet (\forall y)(Fy \supset Gyx)] \supset (Fz \bullet Gzz)\}$	by RP7
	$(\exists z)(\exists x)\{(\forall y)[Fx \bullet (Fy \supset Gyx)] \supset (Fz \bullet Gzz)\}$	by RP4
3.9.33	$(\exists z)(\exists x)(\exists y)\{[Fx \bullet (Fy \supset Gyx)] \supset (Fz \bullet Gzz)\}$	by RP9

In the second set, I will work with the 'x', then the 'y', then the 'z'.

	$(\forall x)\{[Fx \bullet (\forall y)(Fy \supset Gyx)] \supset (\exists z)(Fz \bullet Gzz)\}$	by RP10
	$(\forall x)\{(\forall y)[Fx \bullet (Fy \supset Gyx)] \supset (\exists z)(Fz \bullet Gzz)\}$	by RP4
	$(\forall x)(\exists y)\{[Fx \bullet (Fy \supset Gyx)] \supset (\exists z)(Fz \bullet Gzz)\}$	by RP9
3.9.34	$(\forall x)(\exists y)(\exists z)\{[Fx \bullet (Fy \supset Gyx)] \supset (Fz \bullet Gzz)\}$	by RP7

3.9.33 and 3.9.34 are equivalent to 3.9.32 (and 3.9.31). 3.9.33 and 3.9.34 are both in prenex form. But, they differ in form from each other. There are two other prenex forms equivalent to 3.9.32, which I leave to you as an exercise.

More entailments, and some non-entailments

Let's look at some more entailments and equivalences in predicate logic, and some statements that are not equivalent.

3.9.35 $\vdash$ 3.9.36 but 3.9.36 $\sim\vdash$ 3.9.35.

3.9.35 $(\exists x)[Px \bullet (\forall y)(Qy \supset Rxy)]$

3.9.36 $(\exists x)(\forall y)[Px \supset (Qy \supset Rxy)]$

We can show the entailment using the rules of inference.

3.9.35 $\vdash$ 3.9.36

1. $(\exists x)[Px \bullet (\forall y)(Qy \supset Rxy)]$	
| 2. $\sim(\exists x)(\forall y)[Px \supset (Qy \supset Rxy)]$	AIP
| 3. $(\forall x)(\exists y)\sim[Px \supset (Qy \supset Rxy)]$	2, QE
| 4. $(\forall x)(\exists y)\sim[\sim Px \vee \sim Qy \vee Rxy]$	3, Impl, Impl
| 5. $(\forall x)(\exists y)(Px \bullet Qy \bullet \sim Rxy)$	4, DM, DN
| 6. $Pa \bullet (\forall y)(Qy \supset Ray)$	1, EI
| 7. $(\exists y)(Pa \bullet Qy \bullet \sim Ray)$	5, UI
| 8. $Pa \bullet Qb \bullet \sim Rab$	7, EI
| 9. $(\forall y)(Qy \supset Ray)$	6, Com, Simp
| 10. $Qb \supset Rab$	9, UI
| 11. Qb	8, Com, Simp
| 12. Rab	10, 11, MP
| 13. $\sim Rab$	8, Com, Simp
| 14. $Rab \bullet \sim Rab$	12, 13, Conj
15. $(\exists x)(\forall y)[Px \supset (Qy \supset Rxy)]$	2-14, IP, DN

QED

To see that 3.9.36 $\sim\vdash$ 3.9.35, we can construct a counterexample in a universe with two-members I'll expand 3.9.36 in two steps, first removing the existential quantifier, then the universal.

3.9.36' $(\forall y)[Pa \supset (Qy \supset Ray)] \vee (\forall y)[Pb \supset (Qy \supset Rby)]$

3.9.36" $\{[Pa \supset (Qa \supset Raa)] \bullet [Pa \supset (Qb \supset Rab)]\} \vee \{[Pb \supset (Qa \supset Rba)] \bullet [Pb \supset (Qb \supset Rbb)]\}$

I'll do the same for 3.9.35.

3.9.35' $\quad [Pa \bullet (\forall y)(Qy \supset Ray)] \vee [Pb \bullet (\forall y)(Qy \supset Rby)]$
3.9.35" $\quad [Pa \bullet (Qa \supset Raa) \bullet (Qb \supset Rab)] \vee [Pb \bullet (Qa \supset Rba) \bullet (Qb \supset Rbb)]$

To form the counterexample, assign false to both 'Pa' and 'Pb'. Then, both disjuncts in 3.9.35" are false, but all the conditionals in 3.9.36" are (vacuously) true.

Here are some more entailments and non-entailments in metalogical form. You could demonstrate the entailments, using our system of inference, by considering a specific instance of each. You could prove the non-entailments by instantiating each one and constructing a counterexample, as I did just above.

3.9.37 $\quad (\forall x)Fx \vee (\forall x)Gx \ \vdash \ (\forall x)(Fx \vee Gx)$
3.9.38 $\quad (\forall x)(Fx \vee Gx) \ {\sim}\vdash \ (\forall x)Fx \vee (\forall x)Gx$

To see 3.9.38, just substitute 'P' for F and '~P' for G

3.9.39 $\quad (\exists x)(Fx \bullet Gx) \ \vdash \ (\exists x)Fx \bullet (\exists x)Gx$
3.9.40 $\quad (\exists x)Fx \bullet (\exists x)Gx \ {\sim}\vdash \ (\exists x)(Fx \bullet Gx)$

3.9.41 $\quad (\forall x)(Fx \supset \alpha) \ \vdash \ (\forall x)Fx \supset \alpha$
3.9.42 $\quad (\forall x)Fx \supset \alpha \ {\sim}\vdash \ (\forall x)(Fx \supset \alpha)$

3.9.43 $\quad (\exists x)Fx \supset \alpha \ \vdash \ (\exists x)(Fx \supset \alpha)$ $\quad$ e.g. $(\exists x)Px \supset (\exists y)Qy \ \vdash \ (\exists x)[Px \supset (\exists y)Qy]$
3.9.44 $\quad (\exists x)(Fx \supset \alpha) \ {\sim}\vdash \ (\exists x)Fx \supset \alpha$ $\quad$ e.g. $(\exists x)[Px \supset (\exists y)Qy] \ {\sim}\vdash \ (\exists x)Px \supset (\exists y)Qy$

Exercises 3.9a. Using the rules of passage, transform each formula with a quantifier having narrow scope into one with that quantifier having wide scope.

1. (∃x)(Ax • ~Bx) ∨ (∃x)(Cx ∨ Dx)

2. (∀x)Fx ⊃ (∃x)Gx

3. (∃x)[Hx • (∃y)(Iy • Jxy)]

4. (∀x)Px ⊃ Ra

5. (∃x)[Kx • (∀y)(Ly ⊃ Mxy)]

6. (∃x)Jx ⊃ (∃y)Ky

7. (∃x)(Px • Qx) ⊃ (Pa • Ra)

8. (∃x)[Nx ∨ (∃y)(Oy • Pxy)]

9. (∀x)(Ex ⊃ Fx) • (∀x)(~Fx ≡ Gx)

10. (∀x)[Qx ∨ (∀y)(Ry ⊃ Sxy)]

11. (∀x)[(∀y)Dxy ⊃ Ex]

12. (∀x)[Tx ⊃ (∃y)(Uy • Vxy)]

13. (∀x)[Ax ⊃ (∀y)(By ⊃ Cxy)]

14. (∀x)[(∃y)Rxy ⊃ (Px • Qx)]

Exercises 3.9b. Using the rules of passage, transform each formula with a quantifier having wide scope into one with that quantifier having narrow scope.

1. (∀x)[Rx • (Tx ∨ Sx)]

2. (∃x)(∃y)[(Kx ≡ Lx) • (My • Nxy)]

3. (∃x)(∀y)[(Dx • Ex) ∨ (Fy ⊃ Gxy)]

4. (∀x)(∃y)[Hx ⊃ (Iy • Jxy)]

5. (∃x)[(Ox ∨ Qx) ⊃ (∃y)Py]

6. (∀x)(∀y)[Axy ⊃ (Bx ∨ Cx)]

7. (∀x)(∀y)[(Ax ∨ ~Cx) ⊃ (By ⊃ Dxy)]

8. (∃x)(∀y)[(Fx • Gx) • (Hy ⊃ Exy)]

9. (∀x)(∃y)[Ixy ⊃ (Jx • Kx)]

10. (∃x)[(Lx ≡ Mx) ∨ Nx]

11. (∀x)[(Px • ~Qx) ⊃ (∃y)Oy]

12. (∀x)(∃y)[Sx ⊃ (Ty • Rxy)]

Exercises 3.9c. Translate each of the following sentences into predicate logic.

1. Some math majors take a statistics class and a differential equations class. (Dx, Mx, Sx, Txy: x takes y)

2. Some tigers and all cheetahs are faster than all lions. (Cx, Lx, Tx, Fxy: x is faster than y)

3. All ballet dancers buy tights from some store. (Bx, Sx, Tx, Bxyz: x buys y from z)

4. Some people read all books written by some one. (Bx, Px, Rxy: x reads y, Wxy: x writes y)

5. Honest lawyers are always defeated by dishonest lawyers. (Hx, Lx, Dxy: x defeats y)

6. Some people drive from an office to a home. (Hx, Ox, Px, Dxyz: x drives from y to z)

7. Some people do not drive from an office to a home.

8. Some people drive from a home to an office.

9. No one drives from an office to a home.

10. All sprinters are faster than all cross country runners. (Cx, Sx, Fxy: x is faster than y,)

11. Some robbers steal jewels from the elderly. (Ex, Jx, Rx, Sxyz: x steals y from z)

12. No robber steals jewels from the elderly.

13. Some kind people rescue dogs from some abuse. (Ax, Dx, Kx, Px, Rxyz: x rescues y from z)

14. Some evil burglars kidnap innocent children from homes. (Bx, Cx, Ex, Hx, Ix, Kxyz: x kidnaps y from z)

15. Joel drinks water from a sink. (j, Wx, Sx, Dxyz: x drinks y from z)

16. Some thirsty people drink water from sinks. (Px, Tx, Wx, Sx, Dxyz: x drinks y from z)

17. Aaron eats chocolate from a floor. (a, Cx, Fx, Oxy: x is on y, Exy: x eats y)

18. Some young boys eat chocolate from a floor. (Bx, Cx, Fx, Yx, Oxy: x is on y, Exy: x eats y)

19. All people eat vegetables grown by farmers. (Fx, Px, Vx, Exy: x eats y, Gxy: x grows y)

20. All intelligent students read books written by a professor. (Bx, Ix, Px, Sx, Rxy: x reads y, Wxy: x writes y)

21. Some sweaty people swim in cool lakes. (Cx, Lx, Px, Sx, Sxy: x swims in y)

22. Some rich college students buy expensive textbooks from a bookstore. (Bx, Cx, Ex, Rx, Sx, Tx, Bxyz: x buys y from z)

23. Some impudent girls take warm clothing from their older sisters. (Cx, Gx, Ix, Wx, Oxy: x is older than y, Sxy: x is a sister of y, Txyz: x takes y from z)

24. No impudent girls take clothing from their younger sisters. (Cx, Gx, Ix, Wx, Oxy: x is older than y, Sxy: x is a sister of y, Txyz: x takes y from z)

25. Some wasteful doctors perform unnecessary tests on all gullible clients. (Cx, Dx, Gx, Tx, Ux, Wx, Pxyz: x performs y on z)

26. All carnivorous animals hunt for food. (Ax, Cx, Fx, Hxy: x hunts y)

27. Some men with facial hair get shaved by a town barber. (Bx, Mx, Fx, Tx, Sxy: x shaves y)

28. No men with facial hair get shaved by a town barber.

29. A town barber shaves himself.

30. A town barber shaves everyone in town.

31. A barber in town shaves all men in town who do not shave themselves.

Exercises 3.9d. The rules of passage do not include transformations for the biconditional. Determine the relations among 3.9.45 - 3.9.48.

3.9.45	$(\exists x)(\alpha \equiv Fx)$
3.9.46	$\alpha \equiv (\exists x)Fx$
3.9.47	$(\forall x)(\alpha \equiv Fx)$
3.9.48	$\alpha \equiv (\forall x)Fx$

§3.10: Derivations in **F**

In §3.8, I motivated extending our language **M** to a language **F** by introducing relational predicates to regiment argument 3.8.1.

3.8.1 Andrew is taller than Bob.
Bob is taller than Charles.
For any x, y and z, If x is taller than y and y is taller than z, then x is taller than z.
So, Andrew is taller than Charles.

1. Tab
2. Tbc
3. (∀x)(∀y)(∀z)[(Txy • Tyz) ⊃ Txz] / Tac

To derive the conclusion, we use the same rules of inference we used with **M**. When instantiating, we remove quantifiers one at a time, taking care to make appropriate instantiations to variables or constants. We will need to make only one small adjustment to the rule UG, which I will note shortly.

3.10.1

1.	Tab	
2.	Tbc	
3.	(∀x)(∀y)(∀z)[(Txy • Tyz) ⊃ Txz]	/ Tac
4.	(∀y)(∀z)[(Tay • Tyz) ⊃ Taz]	3, UI
5.	(∀z)[(Tab • Tbz) ⊃ Taz]	4, UI
6.	(Tab • Tbc) ⊃ Tac	5, UI
7.	(Tab • Tbc)	1, 2, Conj
8.	Tac	6, 7, MP

QED

Sometimes, as in 3.10.1, we start our derivations by removing all quantifiers. Sometimes we remove the quantifiers in the middle of the proof, rather than at the beginning, as in 3.10.2.

3.10.2

1.	(∃x)[Hx • (∀y)(Hy ⊃ Lyx)]	/ (∃x)(Hx • Lxx)
2.	Ha • (∀y)(Hy ⊃ Lya)	1, EI
3.	Ha	2, Simp
4.	(∀y)(Hy ⊃ Lya) • Ha	2, Com
5.	(∀y)(Hy ⊃ Lya)	4, Simp
6.	Ha ⊃ Laa	5, UI
7.	Laa	6, 3, MP
8.	Ha • Laa	3, 7, Conj
9.	(∃x)(Hx • Lxx)	8, EG

QED

The Restriction on UG

All of our rules for removing and replacing quantifers work in **F** just as they did in **M**, with only exception. Consider the problematic 3.10.3, beginning with a proposition that can be interpreted as ‘Everything loves something’.

3.10.3	1. $(\forall x)(\exists y)Lxy$	
	2. $(\exists y)Lxy$	1, UI
	3. Lxa	2, EI
	4. $(\forall x)Lxa$	3, UG: but wrong!
	5. $(\exists y)(\forall x)Lxy$	4, EG

Given our interpretation of line 1, line 5 reads, 'There's something that everything loves'. It does not follow from the proposition that everything loves something that there is one thing that everything loves. Imagine we ordered all the things in a circle, and everyone loved just the thing to its left. Line 1 would be true, but line 5 would be false. We should not be able to derive step 5 from step 1.

We can locate the problem in step 4 of 3.10.3. In line 2 we universally instantiated to some random object x. So 'x' could have stood for any object. It retains its universal character, even without a universal quantifier to bind it, and so we are free to UG over x.

Then, in line 3, we existentially instantiated. In existentially instantiating, we gave a name, 'a' to the thing which bore relation L to it, to the thing x loves. Once we gave a name to the thing that x loves, x lost its universal character. It could no longer be anything which loves something. It now must be the thing that loves a. 'x' became as particular an object as 'a' is. So, the generalization at line 4 must be blocked. In other words, variables lose their universal character if they are free when EI is used.

We formulate the resultant restriction on UG as 3.10.4.

3.10.4 **You may never UG on a variable when there's a constant present, and the variable was free when the constant was introduced.**

The restriction on UG debars line 4 of 3.10.3 because 'x' was free in line 3 when 'a' was introduced.

3.10.5 contains an acceptable use of UG in **F**.

3.10.5	1. $(\exists x)(\forall y)[(\exists z)Ayz \supset Ayx]$	
	2. $(\forall y)(\exists z)Ayz$	/ $(\exists x)(\forall y)Ayx$
	3. $(\forall y)[(\exists z)Ayz \supset Aya]$	1, EI
	4. $(\exists z)Ayz \supset Aya$	3, UI
	5. $(\exists z)Ayz$	2, UI
	6. Aya	4, 5, MP
	7. $(\forall y)Aya$	6, UG
	8. $(\exists x)(\forall y)Ayx$	7, EG

QED

Note that at line 7, UG is acceptable because 'y' was not free when 'a' was introduced in line3. The restriction 3.10.4 only applies to UG. All other rules are just as they are in monadic predicate logic.

Accidental binding

When using UG or EG, watch for illicit accidental binding. 3.10.6 contain an instance of accidental binding.

3.10.6	$(Pa \bullet Qa) \supset (Fx \vee Gx)$	
	$(\exists x)[(Px \bullet Qx) \supset (Fx \vee Gx)]$	EG

The first proposition already contains two instances of the variable ‘x’. If you try to quantify over the ‘a’ using EG with the variable ‘x’, you bind the latter two singular terms. That is a legitimate use of existential generalization. As long as the variable ‘x’ was not free when the constant ‘a’ was introduced it retains its universal character. Since it is universal, anything said of ‘x’ holds of everything, so it will hold of the object ‘a’. But the resulting claim may not mean what you want. It is more likely that the desired inference is 3.10.7.

3.10.7 $(Pa \bullet Qa) \supset (Fx \vee Gx)$
$(\exists y)[(Py \bullet Qy) \supset (Fx \vee Gx)]$

In 3.10.7, the latter singular terms, the ‘x’s, remain free. We can still bind them with either a universal quantifier or an existential quantifier, later, as in either of the propositions at 3.10.8.

3.10.8 $(\forall x)(\exists y)[(Py \bullet Qy) \supset (Fx \vee Gx)]$
$(\exists x)(\exists y)[(Py \bullet Qy) \supset (Fx \vee Gx)]$

3.10.9 is a conditional proof using relational predicates. 3.10.10 is a more-complex derivation.

3.10.9
1. $(\forall x)[Ax \supset (\forall y)Bxy]$
2. $(\forall x)[Ax \supset (\exists y)Dyx]$ / $(\forall x)[Ax \supset (\exists y)(Bxy \bullet Dyx)]$
$\mid$ 3. Ax ACP
$\mid$ 4. $Ax \supset (\forall y)By$ 1, UI
$\mid$ 5. $Ax \supset (\exists y)Dyx$ 2, UI
$\mid$ 6. $(\forall y)Bxy$ 4, 3, MP
$\mid$ 7. $(\exists y)Dyx$ 5, 3 MP
$\mid$ 8. Dax 7, EI
$\mid$ 9. Bxa 6, UI
$\mid$ 10. $Bxa \bullet Dax$ 9, 8, Conj
$\mid$ 11. $(\exists y)(Bxy \bullet Dyx)$ 10, EG
12. $Ax \supset (\exists y)(Bxy \bullet Dyx)$ 3-11, CP
13. $(\forall x)[Ax \supset (\exists y)(Bxy \bullet Dyx)]$ 12, UG

QED

3.10.10	1. (∀x)(Wx ⊃ Xx)		
	2. (∀x)[(Yx • Xx) ⊃ Zx]		
	3. (∀x)(∃y)(Yy • Ayx)		
	4. (∀x)(∀y)[(Ayx • Zy) ⊃ Zx]		/(∀x)[(∀y)(Ayx ⊃ Wy) ⊃ Zx]
		5. (∀y)(Ayx ⊃ Wy)	ACP
		6. (∃y)(Yy • Ayx)	3, UI
		7. Ya • Aax	6, EI
		8. Aax ⊃ Wa	5, UI
		9. Aax • Ya	7, Com
		10. Aax	9, Simp
		11. Wa	8,10, MP
		12. Wa ⊃ Xa	1, UI
		13. Xa	12, 11, MP
		14. Ya	7, Simp
		15. Ya • Xa	14, 13, Conj
		16. (Ya • Xa) ⊃ Za	2, UI
		17. Za	16, 15, MP
		18. (∀y)[(Ayx • Zy) ⊃ Zx]	4, UI
		19. (Aax • Za) ⊃ Zx	18, UI
		20. Aax • Za	10, 11, Conj
		21. Zx	19, 20, MP
	22. (∀y)(Ayx ⊃ Wy) ⊃ Zx		5-21, CP
	23. (∀x)[(∀y)(Ayx ⊃ Wy) ⊃ Zx]		22, UG
QED			

Notice that at line 17, you might be tempted to discharge your assumption and finish your CP. But, you wouldn't be able to UG over the 'Za'. We have to UI at line 18, retaining a variable for the predicate 'Z'.

Logical truths

We can use CP and IP to prove logical truths in **F**. Here are four logical truths of **F**.

3.10.11	(∀y)[(∀x)Fx ⊃ Fy]
3.10.12	(∀y)[Fy ⊃ (∃x)Fx]
3.10.13	(∃y)[Fy ⊃ (∀x)Fx]
3.10.14	(∃y)[(∃x)Fx ⊃ Fy]

Note that each one has a similarity to one of the four rules for removing or replacing quantifiers. We can also prove them meta-theoretically, by considering interpretations, in the way that we proved RP10 in §9. 3.10.15 contains a derivation of 3.10.11. I leave derivations of the other three for Exercises 3.10c.

3.10.15

	1. ~(∀y)[(∀x)Fx ⊃ Fy]	AIP
	2. (∃y)~[(∀x)Fx ⊃ Fy]	1, QE
	3. (∃y)~[~(∀x)Fx ∨ Fy]	2, Impl
	4. (∃y)[~~(∀x)Fx • ~Fy]	3, DM
	5. (∃y)[(∀x)Fx • ~Fy]	4, DN
	6. (∀x)Fx • ~Fa	3, DM
	7. (∀x)Fx	6, Simp
	8. Fa	7, UI
	9. ~Fa • (∀x)Fx	6, Com
	10. ~Fa	9, Simp
	11. Fa • ~Fa	8, 10, Conj
12. (∀y)[(∀x)Fx ⊃ Fy]		1-8, IP

QED

Exercises 3.10a. Derive the conclusions of each of the following arguments.

1.
1. (∀x)[(∃y)Bxy ⊃ (Ax ∨ Cx)]
2. (∃z)(~Az • ~Cz) / (∃z)(∀y)~Bzy

2.
1. (∃x)[Qx ∨ (∃y)(Ry • Pxy)]
2. ~(∃x)(Sx ∨ Qx) / (∃z)(∃y)(Ry • Pzy)

3.
1. (∀x)[(∀y)Uxy ⊃ (Tx • Vx)]
2. ~(∃x)Tx / (∃z)~Uza

4.
1. (∃x)[Mx • (∃y)(Ny • Lxy)]
2. (∀x)(∀y)(Lxy ⊃ (∃z)Oyz) / (∃x)(∃y)Oxy

5.
1. Aa • (Ba • ~Cab)
2. (∀y)Cay ∨ (∀z)Dbz / (∃y)(∀z)Dyz

6.
1. (∀x)[Ex • (Fx ∨ Gx)]
2. (∃x){Hx • (∀y)[(Fy ∨ Gy) ⊃ Ixy]} / (∃y)(∃x)Ixy

7.
1. (∀x)[(Fx • Hx) ⊃ (∀y)(Gy • Ixy)]
2. (∃x)[Jx • (∀y)(Gy ⊃ ~Ixy)] / ~(∀z)(Fz • Hz)

8.
1. (∀x)[Ex ⊃ (∀y)(Fy • Gxy)]
2. (∃x)(Ex • Hxb) / (∃x)(∃y)(Gxy • Hxy)

9.
1. (∀x)[Ux ⊃ (∃y)(Ty • Vxy)]
2. (∃x)Vax ⊃ (∀x)Vax
3. Ua / (∃x)(∀y)Vxy

10.
1. (∀x)(∀y)[Ax ⊃ (Dy ⊃ Byx)]
2. (∃x)(∀y)[Dx • (Bxy ⊃ Cy)] / (∀x)Ax ⊃ (∃y)Cy

11. 1. $(\forall x)[Ax \supset (\exists y)(Cy \bullet Dxy)\}$
2. $(\forall x)(\forall y)(Dxy \supset By)$ / $(\forall x)Ax \supset (\exists y)(By \bullet Cy)$

12. 1. $(\exists x)\{Px \bullet (\forall y)[Oy \supset (\forall z)(Rz \supset Qxyz)]\}$
2. $(\forall x)[Px \equiv (Ox \bullet Rx)]$ / $(\exists x)Qxxx$

13. 1. $(\forall x)(Mx \supset {\sim}Ox) \supset (\exists y)Ny$
2. $(\forall y)[Ny \supset (\exists z)(Pz \bullet Qyz)]$
3. ${\sim}(\exists x)(Mx \bullet Ox)$ / $(\exists x)[Nx \bullet (\exists y)Qxy]$

14. 1. $(\forall x)(\forall y)[Kx \supset (My \supset Lxy)]$
2. $(\exists x)(\exists y)[Mx \bullet (Ky \bullet Nxy)]$ / $(\exists y)(\exists x)(Lxy \bullet Nyx)$

15. 1. $(\forall x)[Rx \supset (\forall y)(Ty \supset Uxy)]$
2. $(\forall y)[(\forall x)(Uxy \supset Sy)]$ / $(\forall x)[(Rx \bullet Tx) \supset (\exists y)Sy]$

16. 1. $(\exists x)(\forall y)[(Fx \bullet Dx) \vee (Ey \supset Gxy)]$
2. $(\forall x)[(\exists y)Gxy \supset (\exists z)Hxz]$
3. ${\sim}(\exists x)Fx \bullet (\forall z)Ez$ / $(\exists y)(\exists z)Hyz$

17. 1. $(\forall x)(Kx \equiv Lx) \bullet (\forall x)Jx$
2. $(\forall x)[Jx \supset (\exists y)({\sim}Ky \bullet Mxy)]$ / $(\exists x)({\sim}Lx \bullet Mxx)$

18. 1. $(\forall x)[Kx \supset (\exists y)(Jy \bullet Ixy)]$
2. $(\forall x)(\forall y)(Ixy \supset Lx)$ / $(\forall x)({\sim}Kx \vee Lx)$

19. 1. $(\forall x)[(Ox \supset Nx) \supset (\forall y)(Qy \bullet {\sim}Rxy)]$
2. $(\forall y)(\forall x)(Pxy \supset Rxy)$ / $(\forall x)[(Nx \vee {\sim}Ox) \supset (\forall y){\sim}(Qy \supset Pxy)]$

20. 1. $(\forall x)[(Fx \equiv Hx)$
2. $(\forall x)(Hx \supset {\sim}Ix)$
3. $(\exists x)[Fx \bullet (\exists y)(Iy \bullet {\sim}Gxy)]$ / $(\exists x)[(Fx \bullet {\sim}Ix) \bullet (\exists y)(Iy \bullet {\sim}Gxy)]$

21. 1. $(\forall x)\{Ax \supset (\exists y)[By \bullet (\forall z)({\sim}Cz \bullet Dzxy)]\}$
2. ${\sim}(\forall x)(Ax \supset Cx)$ / $(\exists x)(\exists y)Dxxy$

22. 1. $(\forall x)[(Bx \supset Ax) \supset (\exists y)(Cy \bullet Dxy)]$
2. $(\forall x)[(\forall y){\sim}Dxy \vee Ex]$
3. $(\exists x)Ex \supset {\sim}(\exists x){\sim}Cx$ / $(\forall x)Bx$

23. 1. $(\forall x)\{(Tx \supset {\sim}Sx) \supset (\exists y)[Uy \vee (\forall z)(Vz \supset Wxyz)]\}$
2. ${\sim}(\exists x)(Tx \equiv Sx)$
3. ${\sim}(\exists x)(Vx \supset Ux)$ / $(\exists x)(\exists y)Wxyy$

24. 1. $(\forall x)[Fx \supset (\exists y)(Hy \bullet Gxy)]$
2. $(\forall x)[Hx \supset (\exists y)(Ey \bullet Gxy)]$
3. $(\forall x)[Ex \supset (\forall y)Fy]$ / $(\forall x)Fx \equiv (\exists x)Ex$

25. 1. $(\forall x)\{Jx \supset (\forall y)[My \supset (\forall z)(Lz \supset Kxyz)]\}$
2. $(\exists x)(\exists y)[Mx \bullet (Jy \bullet Nxy)]$
3. $\sim(\forall x)(Lx \supset Ox)$ / $(\exists x)\{Mx \bullet (\exists y)[Nxy \bullet (\exists z)(\sim Oz \bullet Kyxz)]\})$

26. 1. $(\forall x)[Tx \supset (\forall y)(Vy \supset Uxy)]$
2. $\sim(\exists x)(Tx \bullet Sx)$
3. $Ta \bullet Vb$ / $(\exists x)[\sim Sx \bullet (\exists y)Uxy]$

Exercises 3.10b. Translate and derive.

1. There are penguins in an ocean. Some fish swim faster than no penguins. So, some fish does not swim faster than something. (Fx, Px, Ox, Ixy: x is in y, Sxy: x swims faster than y)

2. Some ballet dancers are shorter than some gymnasts. No gymnasts are clumsy. So, it is not the case that all things are clumsy. (Bx, Gx, Cx, Sxy: x is shorter than y)

3. Anyone who teaches a math class is intelligent. Professor Rosen is a person who teaches Calculus I. Calculus I is a math class. So, Professor Rosen is intelligent. (c, r, Px, Ix, Mx, Txy: x teaches y)

4. All cats love all dogs. It is not the case that everything loves Brendan, and all things are cats. So, it is not the case that everything is a dog. (Cx, Dx, b, Lxy: x loves y)

5. All cheetahs are faster than some tigers. Everything is striped if, and only if, it is a tiger. So, if some things are cheetahs, then some things have stripes. (Cx, Tx, Sx, Fxy: x is faster than y)

6. Alice buys a baguette from some store. Baguettes are food. Alice lives in Clinton. So, some residents of Clinton buy some food from some store. (a, c, Bx, Fx, Sx, Lxy: x lives in y, Bxyz: x buys y from z)

7. All philosophers have some mentor to whom they respond. Either something isn't a philosopher or nothing is a mentor. So, not everything is a philosopher. (Mx, Px, Rxy: x responds to y)

8. Some students read books written by professors. All books written by professors are well-researched. So, some professor wrote a well-researched book. (Bx, Px, Sx, Wx, Rxy: x reads y, Wxy: x wrote y)

9. Sunflowers and roses are plants. Some sunflowers grow taller than all roses. Russell gave a rose to Emily. So, some plant is taller than some rose. (e, r, Px, Rx, Sx, Gxy: x grows taller than y, Gxyz: x gives y to z)

10. Something is either expensive if and only if it is of good quality or it is trendy. Some things are meaningful and serve a purpose. All meaningful things are expensive. Nothing is of good quality. So, some trendy things serve a purpose. (Ex, Qx, Tx, Mx, Px)

Exercises 3.10c. Derive the following logical truths of **F**.

1. Derive 3.10.12: $(\forall y)[Fy \supset (\exists x)Fx]$

2. Derive 3.10.13: $(\exists y)[Fy \supset (\forall x)Fx]$

3. Derive 3.10.14: $(\exists y)[(\exists x)Fx \supset Fy]$

4. $(\exists x)(\forall y)Cxy \supset (\forall y)(\exists x)Cxy$

5. $(\forall x)(\exists y)Hxy \supset (\exists x)(\exists y)Hxy$

6. $Fa \vee [(\forall x)Fx \supset Ga]$

7. $(\exists x)Ix \vee (\forall x)(Ix \supset Jx)$

8. $[(\forall x)Dx \vee (\forall x)Ex] \supset (\forall x)(Dx \vee Ex)$

9. $[(\exists x)Ax \supset Ba] \equiv (\forall x)(Ax \supset Ba)$

10. $(\exists x)(Ka \bullet Lx) \equiv [Ka \bullet (\exists x)Lx]$

§3.11: The Identity Predicate: Translation

In this section and the next, we will explore an extension to the system of inference we have adopted for our language **F**. This extension concerns a special two-place relation, identity. In translation, identity allows us to use **F** to regiment a wide range of concepts including some fundamental mathematical concepts. In the next section, we will add some simple rules governing identity which will allow us to make some powerful inferences.

There is some debate about whether identity is strictly a logical relation. I start by explaining that debate, and then proceed, in the remainder of this section, to show how to use identity in translation.

Some claims, like 3.11.1, are paradigmatically logical.

3.11.1 If P then P $P \supset P$

Other claims, like 3.11.2, are paradigmatically non-logical.

3.11.2 It snows in winter in Quebec.

Other claims fall somewhere in between. 3.11.3 is generally not considered a logical truth, even though it has something of the feel of a logical truth.

3.11.3 All bachelors are unmarried.

Philosophers generally characterize the truth of 3.11.3 as semantic, rather than logical. It follows from the meaning of the term 'bachelor' that all bachelors are unmarried. But, that's generally considered a semantic, not a logical, entailment. The line between logical claims and non-logical claims is not always clear, though. Entailments surrounding identity, like the inference at 3.11.4, are generally considered logical.

3.11.4
1. Superman can fly.
2. Superman is Clark Kent.
So, Clark Kent can fly.

Identity, as expressed in the second premise of 3.11.4, is a relation among individuals. We could write it 'Isc'. Thus, we could regiment 3.11.4 as 3.11.5.

3.11.5
1. Fs
2. Isc / Fc

The conclusion of 3.11.5 does not follow from the premises in the system of deduction we have adopted with **F**. But philosophers have observed that identity has special logical properties which facilitate inferences like 3.11.5. Thus, we ordinarily give identity its own symbol, '='.

Identity sentences, like those at 3.11.6, thus look a little different from other dyadic relations.

3.11.6
Clark Kent is Superman $c=s$
Mary Ann Evans is George Eliot $m=g$

We do not extended our language **F** by introducing the identity predicate. We only set aside a particular two-place predicate, adding a new shorthand for it. We do not need any new formation rules, though we should clarify how the shorthand works. In particular, formulas like 'a=b' are really short for

'Iab', though we insist that 'Ixy' is interpreted as the identity relation. Since we do not put brackets around 'Iab', we should not put brackets around 'a=b' either. As far as the logical language is concerned, identities are just special kinds of two-place relations.

Negations of identity claims, strictly speaking, are written just like the negations of any other two-place relation, with a tilde in front. Negation applies to the identity predicate, and not to the objects related by that predicate. Since we are using a special symbol for the identity relation, we sometimes write them in shorthand. Both ways of writing negations are displayed at 3.11.7.

3.11.7 $\sim a=b$
$a \neq b$

While the identity predicate needs no new syntactical rules, we will introduce new derivation rules governing the predicate. Technically, we are introducing a new deductive system which uses the same language **F**. There three new rules governing the identity predicate.

For any singular terms α and β:

IDr (reflexivity)	$\alpha=\alpha$
IDs (symmetry)	$\alpha=\beta \rightleftharpoons \beta=\alpha$
IDi (indiscernibility of identicals)	$\mathscr{F}\alpha$ $\alpha=\beta \quad / \mathscr{F}\beta$

IDr says that any singular term stands for an object which is identical to itself. We can add, in any proof, a statement of that form. IDs says that identity is commutative. IDi is the most useful of the three identity rules. Consider again Superman and Clark Kent. We know that the two people are the same, so anything true of one, is true of the other. This property is called Leibniz's law, or the indiscernibility of identicals. IDi says that if you have α=β, then, you may rewrite any formula containing α with β in the place of α throughout.

Be careful not to confuse two related claims. The indiscernibility of identicals, written as a single schematic sentence at 3.11.8, says that two identical objects share all their properties. This claim is not contentious. It might be better cast as: if two terms refer to the same object, then whatever we predicate of the one term can just as easily be predicated of the other. The related but different claim is Leibniz's identity of indiscernibles, written at 3.11.9. This less-plausible claim, which relies on the Principle of Sufficient Reason, says that no two things share all properties.

3.11.8	$(\forall x)(\forall y)[x=y \supset (\mathscr{F}x \equiv \mathscr{F}y)]$	indiscernibility of identicals
3.11.9	$(\forall x)(\forall y)[(\mathscr{F}x \equiv \mathscr{F}y) \supset x=y]$	identity of indiscernibles

I will discuss the special inferential properties of the identity predicate in the next section. In this section, I focus on translation. The identity predicate allows us to reveal inferential structure for a wide variety of propositions, making it extraordinarily powerful.

To start, note that, as a convention, for the rest of the chapter, I will drop the requirement on wffs that series of conjunctions and series of disjunctions have brackets for every two conjuncts or disjuncts. Propositions using identity can become long and complex. To reduce the number of brackets in our formulas, given that commutativity and association hold of both conjunction and disjunction, we allow such series, even if they have many terms, to be collected with one set of brackets.

Thus, 3.11.10 can be written as 3.11.11 and 3.11.12 can be written as 3.11.13

3.11.10 $(\exists x)(\exists y)\{(Ax \bullet Bxj) \bullet [(Ay \bullet Iyj) \bullet x \neq y)]\}$
3.11.11 $(\exists x)(\exists y)(Ax \bullet Bxj \bullet Ay \bullet Iyj \bullet x \neq y)$
3.11.12 $(\forall x)(\forall y)(\forall z)(\forall w)\{[(Px \bullet Py) \bullet (Pz \bullet Pw)] \supset \{[(x=y \vee x=z) \vee (x=w \vee y=z)] \vee (y=w \vee z=w)\}\}$
3.11.13 $(\forall x)(\forall y)(\forall z)(\forall w)[(Px \bullet Py \bullet Pz \bullet Pw) \supset (x=y \vee x=z \vee x=w \vee y=z \vee y=w \vee z=w)]$

As we have seen, simple identity claims are easily written, as in 3.11.6. Ordinarily, we think of such claims as holding between two names of a single object.

Statements using terms like‘except’ and ‘only’ can be regimented usefully. To say that John only loves Mary, we add to the claim that John loves Mary (at 3.11.14) the claim that anyone John loves must be identical to Mary, as at 3.11.15. To say that only John loves, Mary, we add the claim that anyone who loves Mary must be identical to John.

3.11.14 John loves Mary Ljm
3.11.15 John only loves Mary $Ljm \bullet (\forall x)(Ljx \supset x=m)$
3.11.16 Only John loves Mary $Ljm \bullet (\forall x)(Lxm \supset x=j)$

3.11.17 contains more model translations using ‘only’.

3.11.17 Nietzsche respects only Spinoza
$Rns \bullet (\forall x)(Rnx \supset x=s)$

Only Nietzsche doesn’t like Nietzsche.
$\sim Lnn \bullet (\forall x)(\sim Lxn \supset x=n)$

Only Locke plays billiards with some rationalist who is read more widely than Descartes.
$(\exists x)\{(Rx \bullet Pxl) \bullet (\forall y)[(Ry \bullet Myd) \supset y=l]\}$

Only Kant is read more widely than Descartes and Hume.
$Mkd \bullet Mkh \bullet (\forall x)[(Mxd \vee Mxh) \supset x=k]$

Sentences with ‘except’ are usually universal claims. We merely add a clause to the antecedent of the conditional which is the main operator in the scope of the univeral quantifier, omitting the desired exception. In 3.11.18, we are saying that John doesn’t love Mary, but every other person does.

3.11.18 Everyone except John loves Mary $\sim Ljm \bullet (\forall x)[(Px \bullet x \neq j) \supset Lxm]$

Ordinarily, when we use ‘except, not only do we exempt one individual from a universal claim, we also deny that whatever we are ascribing to everyone else holds of the exemption. These denials can be seen in the beginnings of the ‘except’ sentences at 3.11.19. Notice that some uses of ‘but’ work just like ordinary uses of ‘except’.

3.11.19 Every philosopher except Berkeley respects Locke

$Pb \bullet {\sim}Rbl \bullet (\forall x)[(Px \bullet x \neq b) \supset Rxl]$

Nietzsche does not respect any philosopher except Spinoza.

$Ps \bullet Rns \bullet (\forall x)[(Px \bullet x \neq s) \supset {\sim}Rnx]$

Some philosopher likes all philosophers except Plato and Aristotle.

$Pp \bullet Pa \bullet (\exists x)\{Px \bullet (\forall y)[(Py \bullet y \neq p \bullet y \neq a) \supset Lxy]\}$

Every philosopher but Socrates wrote a book.

$Ps \bullet {\sim}(\exists x)(Bx \bullet Wtx) \bullet (\forall x)[(Px \bullet x \neq t) \supset (\exists y)(By \bullet Wxy)]$

Relational predicates allowed us to express comparisons: larger than, smaller than, older than, more funny than. The identity predicate allows us to express superlatives. To move from a comparison to a superlative, as at 3.11.20 where Bxy stands for ‘x is a better impressionist than y’, you add a universal clause: better (or more profound or nicer) than anyone. So far, we don’t need identity. But if you are nicer than anyone, then you are nicer than yourself, which is impossible. So, we need identity to rule out the single, reflexive case.

3.11.20 Degas is a better impressionist than Monet $Id \bullet Im \bullet Bdm$

Degas is the best impressionist $Id \bullet (\forall x)[(Ix \bullet x \neq d) \supset Bdx]$

3.11.21 has more superlatives.

3.11.21 Hume is the biggest philosopher.

$Ph \bullet (\forall x)[(Px \bullet x \neq h) \supset Bhx]$

Hume is not the most difficult empiricist to read.

$Eh \bullet {\sim}(\forall x)[(Ex \bullet x \neq h) \supset Dxh]$

The Ethics is the most difficult book by Spinoza to read.

$Be \bullet Wse \bullet (\forall x)[(Bx \bullet Wsx \bullet x \neq e) \supset Dex]$

Either *The Critique of Pure Reason* or *The Ethics* is the most difficult book to read.

$Bc \bullet Be \bullet (\forall x)[(Bx \bullet x \neq c \bullet x \neq e) \supset (Dcx \vee Dex)]$

The last two uses of identity that I will discuss are especially philosophically interesting. The first concerns how much mathematics can be developed using just logic. The latter concerns a puzzle in the philosophy of language, often called the problem of empty reference.

Frege’s development of formal logic was intricately linked to his logicist project of trying to show that mathematics is just logic in complex form. Frege’s logicism, as he developed it, was a failure; he used an inconsistent logic. Subsequent logicist projects are forced to rely on substantial set-theoretic principles that many philosophers believe are not strictly logical. Normally, we extend logical systems to mathematical ones by including one more element to the language, ‘$\in$’, standing for set inclusion, and axioms governing set theory. Mathematics is uncontroversially reducible to logic plus set theory.

Some contemporary logicians continue to work on logicism; they are ordinarily known as neo-logicists, or neo-Fregeans. Part of the contemporary neo-logicist project is to see just how little set theory we need to add to logic in order to develop mathematics. It is edifying to see, then, how much mathematics can be generated by the logical machinery of just **F**, using the identity predicate.

For example, we can express many adjectival uses of numbers in **F**. We have already seen how to say that there is one of something, using 'only' as in 3.11.22.

3.11.22 There is only one aardvark. $(\exists x)[Ax \bullet (\forall y)(Ay \supset x=y)]$

We could rephrase 3.11.22 as 'there is exactly one aardvark'. So, we have already seen how to translate sentences including 'exactly one' clauses. To regiment 'exactly' sentences for larger numbers, to say that there are exactly n of some object, we combine at-least and at-most clauses. Let's start with some at-least sentences, as at 3.11.23. Notice that there is a natural procedure for translating 'at least' for any number. The identity predicate is used to make sure that each of the quantifiers refers to a distinct individual.

3.11.23 There is at least one aardvark. $(\exists x)Ax$

There are at least two aardvarks. $(\exists x)(\exists y)(Ax \bullet Ay \bullet x \neq y)$

There are at least three aardvarks.
$(\exists x)(\exists y)(\exists z)(Ax \bullet Ay \bullet Az \bullet x \neq y \bullet x \neq z \bullet y \neq z)$

There are at least four aardvarks.
$(\exists x)(\exists y)(\exists z)(\exists w)(Ax \bullet Ay \bullet Az \bullet Aw\ x \neq y \bullet x \neq z \bullet x \neq w \bullet y \neq z \bullet y \neq w \bullet z \neq w)$

The identity clauses at the end become increasingly long as the number we are expressing increases, but the algorithm is simple: just make sure to include one clause for each pair of variables. 3.11.24 contains more at-least sentences.

3.11.24 At least one materialist respects Berkeley. $(\exists x)(Mx \bullet Rxb)$

At least two materialists respect Berkeley.
$(\exists x)(\exists y)(Mx \bullet Rxb \bullet My \bullet Ryb \bullet x \neq y)$

There are at least three materialists who respect Berkeley.
$(\exists x)(\exists y)(\exists z)(Mx \bullet Rxb \bullet My \bullet Ryb \bullet Mz \bullet Rzb \bullet x \neq y \bullet x \neq z \bullet y \neq z)$

At least two idealist philosophers respect each other.
$(\exists x)(\exists y)(Ix \bullet Px \bullet Iy \bullet Py \bullet Rxy \bullet Ryx \bullet x \neq y)$

At least three coherentists respect some book by Descartes.
$(\exists x)(\exists y)(\exists z)\{Cx \bullet Cy \bullet Cz \bullet (\exists w)[(Bw \bullet Wdw) \bullet Rxw] \bullet (\exists w)[(Bw \bullet Wdw) \bullet Ryw] \bullet (\exists w)[(Bw \bullet Wdw) \bullet Rzw] \bullet x \neq y \bullet x \neq z \neq y \neq z\}$

At-most clauses use universal quantifiers. The core idea is that to say that one has at most n of something, we say that if we think we have one more than n of it, there must be some redundancy.

3.11.25 There is at most one aardvark. $(\forall x)(\forall y)[(Ax \bullet Ay) \supset x=y]$

There are at most two aardvarks.
$(\forall x)(\forall y)(\forall z)[(Ax \bullet Ay \bullet Az) \supset (x=y \vee x=z \vee y=z)]$

There are at most three aardvarks.

$(\forall x)(\forall y)(\forall z)(\forall w)[(Ax \bullet Ay \bullet Az \bullet Aw) \supset (x=y \vee x=z \vee x=w \vee y=z \vee y=w \vee z=w)]$

As with at-least sentences, we have identity clauses at the end. But for at-most sentences, the identity clauses are affirmative and we disjoin them. Again, make sure to have one clause for each pair of variables.

3.11.26 Nietzsche respects at most one philosopher.

$(\forall x)(\forall y)[(Px \bullet Rnx \bullet Py \bullet Rny) \supset x=y]$

Nietzsche respects at most two philosophers.

$(\forall x)(\forall y)(\forall z)[(Px \bullet Rnx \bullet Py \bullet Rny \bullet Pz \bullet Rnz) \supset (x=y \vee x=z \vee y=z)]$

Kant likes at most two empiricists better than Hume.

$(\forall x)(\forall y)(\forall z)[(Ex \bullet Lkxh \bullet Ey \bullet Lkyh \bullet Ez \bullet Lkzh) \supset (x=y \vee x=z \vee y=z)]$

At most one idealist plays billiards with some rationalist.

$(\forall x)(\forall y)\{Ix \bullet (\exists z)(Rz \bullet Pxz) \bullet Iy \bullet (\exists z)(Rz \bullet Pyz)] \supset x=y\}$

At most two rationalists wrote a book more widely read than every book written by Hume.

$(\forall x)(\forall y)(\forall z)\{\{Rx \bullet (\exists w)[Bw \bullet Wxw \bullet (\forall z)(Bz \bullet Whz) \supset Mwz] \bullet Ry \bullet (\exists w)[Bw \bullet Wyw \bullet (\forall z)(Bz \bullet Whz) \supset Mwz] \bullet Rz \bullet (\exists w)[Bw \bullet Wzw \bullet (\forall z)(Bz \bullet Whz) \supset Mwz]\} \supset (x=y \vee x=z \vee y=z)\}$

To express ‘exactly, we combine the at-least and at-most clauses. 3.11.22 says that there is exactly one aardvark. The first portion says that there is at least one. The second portion, starting with the universal quantifier, expresses the redundancy which follows from supposing that there are two aardvarks. We still need n+1 quantifiers in an ‘exactly’ sentence, but the first n quantifiers are existential; we need only one further universal quantifier. The identity clauses at the end of the at-most section only hold between the variable bound by the universal quantifier and the other variables, not among the existentially-bound variables.

3.11.27 There are exactly two aardvarks.

$(\exists x)(\exists y)\{Ax \bullet Ay \bullet x \neq y \bullet (\forall z)[Az \supset (z=x \vee z=y)]\}$

There are exactly three aardvarks.

$(\exists x)(\exists y)(\exists z)\{Ax \bullet Ay \bullet Az \bullet x \neq y \bullet x \neq z \bullet y \neq z \bullet (\forall w)[Aw \supset (w=x \vee w=y \vee w=z)]\}$

There are exactly four aardvarks.

$(\exists x)(\exists y)(\exists z)(\exists w)\{Ax \bullet Ay \bullet Az \bullet Aw \bullet x \neq y \bullet x \neq z \bullet x \neq w \bullet y \neq z \bullet y \neq w \bullet z \neq w \bullet (\forall v)[Av \supset (v=x \vee v=y \vee v=z \vee v=w)]\}$

These numerical sentences get very long very quickly. Indeed, **F** can not express ‘exactly five’ or more, since we have run out of quantifiers. To abbreviate numerical sentences, logicians sometimes introduce special shorthand quantifiers like the ones at 3.11.28.

3.11.28 $(\exists 1x), (\exists 2x), (\exists 3x)$...

Sometimes the quantifiers at 3.11.28 are taken to indicate that there are at least the number indicated. To indicate exactly a number, '!' is sometimes used. For exactly one thing, people sometimes write '$(\exists !x)$'. For more things, we can insert the number and the '!', as at 3.11.29.

3.11.29 $(\exists 1!x), (\exists 2!x), (\exists 3!x)$...

These abbreviations are useful for translation. But once we want to make inferences using the numbers, we have to unpack their longer forms. We will not extend our language **F** to include more variables, or to include numerals or '!', but it is easy enough to do so. 3.11.30 contains further 'exactly' translations.

3.11.30 There are exactly two chipmunks in the yard.

$$(\exists x)(\exists y)\{Cx \bullet Yx \bullet Cy \bullet Yy \bullet x \neq y \bullet (\forall z)[(Cz \bullet Yz) \supset (z=x \vee z=y)]\}$$

There is exactly one even prime number.

$$(\exists x)\{(Ex \bullet Px \bullet Nx) \bullet (\forall y)[(Ey \bullet Py \bullet Ny) \supset y=x]\}$$

There are exactly three aardvarks on the log.

$$(\exists x)(\exists y)(\exists z)\{Ax \bullet Lx \bullet Ay \bullet Ly \bullet Az \bullet Lz \bullet x \neq y \bullet \bullet x \neq z \bullet y \neq z \bullet (\forall w)[(Aw \bullet Lw) \supset (w=x \vee w=y \vee w=z]\}$$

Our last use of the identity predicate is in a solution to a problem in the philosophy of language. The problem can be seen in trying to interpret 3.11.31.

3.11.31 The king of America is bald.

We might regiment 3.11.27 as 3.11.28.

3.11.32 Bk

3.11.32 is false, since there is no king of America. Given our bivalent semantics, 3.11.33 should be true since it is the negation of a false statement.

3.11.33 ~Bk

3.11.33 seems to be a perfectly reasonable regimentation of 3.11.34.

3.11.34 The king of America is not bald.

3.11.34 has the same grammatical form as 3.11.35.

3.11.35 Devendra Banhart is not bald.

3.11.35 entails that Devendra Banhart has hair. So, 3.11.34 may reasonably be taken to imply that the king of America has hair. But, we don't want to make that inference.

In fact, we want both 3.11.32 and 3.11.33 to be false. But, the conjunction of their negations is the contradiction 3.11.36.

3.11.36 $\sim$Bk • $\sim\sim$Bk

We had better regiment 3.11.31 and 3.11.34 differently.

Bertrand Russell, facing just this problem, focused on the fact that 'the king of America' is a definite description which refers to no real thing. There are two ways to refer to an object. We can use the name of the object, or we can describe it (e.g. the person who..., the thing that...). Definite descriptions refer to specific objects without using names.

Both 3.11.31 and 3.11.34 use definite descriptions to refer to an object. They are both false due to a false presupposition in the description. Russell claims that definite descriptions are disguised complex propositions. He urges us to unpack them to reveal their true logical form. According to Russell, 3.11.31, properly understood, consists of three simpler expressions.

3.11.31A	There is a king of America.	$(\exists x)Kx$
3.11.31B	There is only one king of America.	$(\forall y)(Ky \supset y=x)$
3.11.31C	That thing is bald.	Bx

Putting them together, so that every term is within the scope of the original existential quantifier, we get 3.11.37, which Russell claims is the proper analysis of 3.11.31.

3.11.37 $(\exists x)[Kx \bullet (\forall y)(Ky \supset y=x) \bullet Bx]$

3.11.31 is false because clause A is false. 3.11.34 is also false, for the same reason, which we can see in its proper regimentation, 3.11.38.

3.11.38 $(\exists x)[Kx \bullet (\forall y)(Ky \supset y=x) \bullet \sim Bx]$

The tilde in 3.1.38 only affects the third clause. The first clause is the same in 3.11.37 and 3.11.38, and still false. Further, when we conjoin 3.11.37 and 3.11.38, we do not get a contradiction, as we did in 3.11.36.

3.11.39 $(\exists x)[Kx \bullet (\forall y)(Ky \supset y=x) \bullet Bx] \bullet (\exists x)[Kx \bullet (\forall y)(Ky \supset y=x) \bullet \sim Bx]$

3.11.39 is no more problematic than 3.11.40.

3.11.40 Some things are purple, and some things are not purple.
$(\exists x)Px \bullet (\exists x)\sim Px$

One might worry that 3.11.37 and 3.11.38 are still problematic, since their uniqueness clauses seem to make it the case that we are talking about the same thing both having the property of baldness and lacking that property. Let's see why this is not so.

First, note that the problem arises only because we want to assert the negations of 3.11.31 and 3.11.34, and the simple regimentation leads to the contradiction at 3.11.36. If we were to assert both 3.11.37 and 3.11.38, instead of their negations, we would be able to derive a contradiction. But, the contradiction would be present in both 3.11.31 and 3.11.34, too. It is no error in a logic if it derives a contradiction from contradictory statements! So, let's look at the negations of the 3.11.37 and 3.11.38.

3.11.37' $\sim(\exists x)[Kx \bullet (\forall y)(Ky \supset y=x) \bullet Bx]$
3.11.38' $\sim(\exists x)[Kx \bullet (\forall y)(Ky \supset y=x) \bullet \sim Bx]$

Now, let's unpack 3.11.37 and 3.11.38, and see if we can get to a contradiction between them. I'll exchange quantifers, so we have universals, and bring the tildes inside. 3.11.41 starts with 3.11.37' and 3.11.42 starts with 3.11.38'.

3.11.41	$\sim(\exists x)[Kx \bullet (\forall y)(Ky \supset y{=}x) \bullet Bx]$	3.11.37'
	$(\forall x)\sim[Kx \bullet (\forall y)(Ky \supset y{=}x) \bullet Bx]$	QE
	$(\forall x)[\sim Kx \vee \sim(\forall y)(Ky \supset y{=}x) \vee \sim Bx]$	DM
	$(\forall x)[\sim Kx \vee (\exists y)\sim(Ky \supset y{=}x) \vee \sim Bx]$	QE
	$(\forall x)[\sim Kx \vee (\exists y)\sim(\sim Ky \vee y{=}x) \vee \sim Bx]$	Impl
	$(\forall x)[\sim Kx \vee (\exists y)(\sim\sim Ky \bullet \sim y{=}x) \vee \sim Bx]$	DM
	$(\forall x)[\sim Kx \vee (\exists y)(Ky \bullet \sim y{=}x) \vee \sim Bx]$	DN

3.11.42	$\sim(\exists x)[Kx \bullet (\forall y)(Ky \supset y{=}x) \bullet \sim Bx]$	3.11.38'
	$(\forall x)\sim[Kx \bullet (\forall y)(Ky \supset y{=}x) \bullet \sim Bx]$	QE
	$(\forall x)[\sim Kx \vee \sim(\forall y)(Ky \supset y{=}x) \vee Bx]$	DM, DN
	$(\forall x)[\sim Kx \vee (\exists y)\sim(Ky \supset y{=}x) \vee Bx]$	QE
	$(\forall x)[\sim Kx \vee (\exists y)\sim(\sim Ky \vee y{=}x) \vee Bx]$	Impl
	$(\forall x)[\sim Kx \vee (\exists y)(Ky \bullet \sim y{=}x) \vee Bx]$	DM, DN

The conjunction of the last formulas in 3.11.41 and 3.11.42 will not lead to contradiction, even if we instantiate both to the same constant and combine them.

3.11.43	$\sim Ka \vee (\exists y)(Ky \bullet \sim y{=}a) \vee \sim Ba$	3.11.41, UI
	$\sim Ka \vee (\exists y)(Ky \bullet \sim y{=}a) \vee Ba$	3.11.42, UI
	$\{\sim Ka \vee (\exists y)(Ky \bullet \sim y{=}a) \vee \sim Ba\} \bullet \{\sim Ka \vee (\exists y)(Ky \bullet \sim y{=}a) \vee \sim Ba\}$	Conj
	$\sim Ka \vee (\exists y)(Ky \bullet \sim y{=}a) \vee (Ba \bullet \sim Ba)$	Dist

Thus, by asserting both 3.11.37' and 3.11.38', we are asserting only either that there is no king of America, or that there is more than one king of America, or that some thing is both bald and not bald.

Let's put away the problem of empty reference for definite descriptions and see how Russell's analysis guides translation generally. We regiment sentences of the form of 3.11.44 as sentences like 3.11.45. 3.11.46 uses Russell's original example.

3.11.44	The country called a sub-continent is India.
3.11.44A	There is a country called a sub-continent.
3.11.44B	There is only one such country.
3.11.44C	That country is identical with India.

3.11.45 $(\exists x)\{(Cx \bullet Sx) \bullet (\forall y)[(Cy \bullet Sy) \supset y{=}x] \bullet x{=}i\}$

3.11.46 The author of Waverly was a genius.
$(\exists x)\{Wx \bullet (\forall y)[Wy \supset y{=}x] \bullet Gx\}$

Exercises 3.11. Translate into first-order logic, using the identity predicate where applicable.

1. Andre is the busiest student in Logic. (a, l, Sx, Bxy: x is busier than y, Ixy: x is in y)

2. Carlos invites everyone except Belinda. (b, c, Px, Ixy: x invites y)

3. There are at least two speakers at the conference. (Sx)

4. There exactly two speakers at the conference.

5. There is exactly one male honors student. (Hx, Mx, Sx)

6. There are exactly two male honors students.

7. The valedictorian is Diego. (d, Vx)

8. At most one person who attends Riverdale High goes to Harvard. (h, r, Px, Axy: x attends y, Gxy: x goes to y)

9. At most two people who attend Riverdale High go to Harvard.

10. At most three people who attend Riverdale High go to Harvard.

11. Only Carla trains dogs. (c, Dx, Txy: x trains y)

12. Carla only trains dogs.

13. There are at least two newspapers in Jamal's Newsstand. (j, Nx, Ixy: x is in y)

14. There are exactly two newspapers in Jamal's Newsstand.

15. There are exactly three newspapers in Jamal's Newsstand.

16. There are at most two children in Marie's Toy Store. (m, Cx, Ixy: x is in y)

17. There are exactly two children in Marie's Toy Store.

18. Marie's is the biggest store in San Sebastian. (m, s, Sx, Bxy: x is bigger than y, Ixy: x is in y)

19. Of the stores in town, only Gianni's sells ice cream. (g, Ix, Sx, Sxy: x sells y)

20. Everyone except Emilia shops at Gianni's. (e, g, Px, Sxy: x shops at y)

21. The owner of Gianni's is rich. (g, Rx, Oxy: x owns y)

22. There are at least three workers stronger than Fernando. (f, Wx, Sxy: x is stronger than y)

23. The best worker is Fernando. (f, Wx, Bxy: x is a better worker than y)

24. Geraldo is the strongest worker in Canada. (c, g, Wx, Ixy: x is in y, Sxy: x is stronger than y)

25. Only Geraldo lifts heavy things. (g, Hx, Lxy: x lifts y)

26. Everyone except Hector likes Geraldo. (g, h, Px, Lxy: x likes y)

27. At least two workers earn more than Igor. (i, Wx, Exy: x earns more than y)

28. Exactly two workers earn more than Igor.

29. The state called the Big Apple is New York. (n, Ax, Sx)

30. At most two students are from New York. (n, Sx, Fxy: x is from y)

31. At most one New Yorker has an apartment bigger than an apartment of mine. (m, Ax, Nx, Bxy: x is bigger than y, Hxy: x has y)

32. At most two New Yorkers have an apartment bigger than mine.

33. The Empire State Building is taller than the Chrysler Building. (c, e, Txy: x is taller than y)

34. The Empire State Building is the tallest building in New York City. (e, n, Bx, Ixy: x is in y, Txy: x is taller than y)

35. Everyone except Katalin has been to New York. (k, n, Px, Hxy: x has been to y)

36. Everyone except Katalin and Alice have been to New York. (a, k, n, Px, Hxy: x has been to y)

37. There are at least two people shorter than Louisa. (l, Px, Sxy: x is shorter than y)

38. There are at least three people shorter than Louisa.

39. There is exactly one talented singer in Potsdam . (p, Sx, Tx, Ixy: x is in y)

40. The lead singer is famous. (Fx, Lx)

41. *The Departed* is the only film for which Scorsese won an Academy Award (d, s, Ax, Fx, Wxyz: x won y for z)

42. Exactly one talented singer wins *American Idol*. (a, Sx, Tx, Wxy: x wins y)

43. Everyone except Lara watches American Idol. (a, l, Px, Wxy: x watches y)

44. At least three squirrels eat blueberries. (Bx, Sx, Exy: x eats y)

45. At most two squirrels are in a tree. (Sx, Tx, Ixy: x is in y)

46. The Queen of England is British. (Bx, Qx)

47. Queen Elizabeth is the most powerful woman in Britain. (b, e, Wx, Ixy; x is in y, Pxy: x is more powerful than y)

48. Queen Elizabeth has exactly two sons. (e, Sxy: x is the son of y)

49. There is at most one queen in England. (e, Qx, Ixy: x is in y)

50. There is one and only one captain on a ship. (Cx, Sx, Oxy: x is on y)

51. The Titanic is the biggest ship. (t, Sx, Bxy: x is bigger than y)

52. There are at least two logicians who read a paper written by Tarski. (l, t, Px, Sx, Ixy: x is in y, Rxy: x reads y; Wxy: x wrote y)

53. Every student in Logic reads a paper written by Tarski except Mario. (l, m, t, Px, Sx, Ixy: x is in y, Rxy: x reads y; Wxy: x wrote y)

54. Only Nicola and Rick received a higher grade than Juan. (j, n, r, Gx, Hxy: x is higher than y, Rxy: x received y)

55. Every biology major except Petra takes Chemistry 240. (c, p, Bx, Txy: x takes y)

56. There are at least two science majors in Poetry 101. (p, Sx, Ixy: x is in y)

57. There are at least three science majors in Poetry 101.

58. There are at most three psychology majors in Psych 210. (p, Px, Ixy: x is in y)

59. Sandra is the professor who has the biggest class. (s, Cx, Px, Bxy: x is bigger than y, Hxy: x has y)

60. There are exactly two professors who teach Logic. (l, Px, Txy: x teaches y)

§3.12: The Identity Predicate: Derivations

We saw that there are three rules governing the identity predicate. For any singular terms, α and β:

IDr (reflexivity)	$\alpha=\alpha$	
IDs (symmetry)	$\alpha=\beta \rightleftarrows \beta=\alpha$	
IDi (indiscernibility of identicals)	$\mathscr{F}\alpha$	
	$\alpha=\beta$	$/ \mathscr{F}\beta$

IDr is an axiom schema. While we are not generally using an axiomatic system of inference, it is traditional to allow IDr as an axiom. We can add an instance of the axiom schema into any proof, with no line justification. IDs is a rule of equivalence. We can use IDs on whole lines or on parts of lines. IDi is a rule of inference. With IDi, we always re-write a whole line, switching one constant for another.

Let's see some examples of how to use these rules. I'll start with 3.11.4, the inference with which I motivated identity theory.

3.11.4 Superman can fly.
Superman is Clark Kent. So, Clark Kent can fly.

To derive the conclusion, we need only a simple application of IDi.

3.12.1
1. Fs
2. s=c / Fc
3. Fc 1, 2, IDi
QED

3.12.2 uses IDs and IDi.

3.12.2
1. a=b ⊃ j=k
2. b=a
3. Fj / Fk
4. a=b 2, IDs
5. j=k 1, 4, MP
6. Fk 3, 5, IDi
QED

To derive the negation of an identity statement, one often uses IP as in 3.12.3.

3.12.3
1. Rm
2. ~Rj / m≠j
| 3. m=j AIP
| 4. Rj 1, 3, IDi
| 5. Rj • ~Rj 4, 2, Conj
6. m≠j 3-5, IP
QED

3.12.4 uses the reflexivity rule to set up a use of MP.

3.12.4

	1. $(\forall x)(\sim Gx \supset x \neq d)$	/ Gd
	│2. $\sim Gd$	AIP
	│3. $\sim Gd \supset d \neq d$	1, UI
	│4. $d=d$	IDr
	│5. $d \neq d$	3, 2, MP
	│6. $d=d \bullet d \neq d$	4, 5, Conj
	7. Gd	2-6, CP
QED		

Identity statements, recall, are just two-place relations. We can EG over variables in identity statements, as in 3.12.5. Notice the use of IDs at line 5; it works like commutativity for singular terms.

3.12.5

	1. Rab	
	2. $(\exists x)\sim Rxb$	/ $(\exists x)\sim x=a$
	3. $\sim Rcb$	2, EI
	│4. $c=a$	AIP
	│5. $a=c$	4, IDs
	│6. Rcb	1, 5, IDi
	│7. $Rcb \bullet \sim Rcb$	6, 3, Conj
	8. $\sim c=a$	4-7, IP
	9. $(\exists x)\sim x=a$	8, EG
QED		

The derivations 3.12.1 - 3.12.5 have been quick. But, many simple arguments using identity require long derivations. The argument 3.12.6 is valid. It may seem a little odd, since it derives a universal conclusion from an existential premise. But remember that a definite description is definite; there is only one thing that fits the description. The universality of the conclusion is supported by the uniqueness clause in the definite description. The premise entails that there is only one Joyce scholar at Hamilton College. Anything we say of a Joyce scholar at Hamilton holds of all Joyce scholars at Hamilton (viz. only the one).

3.12.6 The Joyce scholar at Hamilton College is erudite. Therefore, all Joyce scholars at Hamilton College are erudite.

$(\exists x)\{Jx \bullet Hx \bullet (\forall y)[(Jy \bullet Hy) \supset x=y] \bullet Ex\}$ / $(\forall x)[(Jx \bullet Hx) \supset Ex]$

As I noted in §3.11, by convention we may drop brackets from series of conjunctions or disjunctions. Given our convention about dropping brackets among series of conjunctions and series of disjunctions, we should add corresponding conventions governing inferences.

If a wff is just a series of conjunctions, you may use Simp to infer, immediately, any of the conjuncts, including multiple conjuncts.
If a wff is just a series of disjunctions, and you have the negation of one of the disjuncts on a separate line, you may eliminate it, using DS, from the series.
You may use Conj to conjoin any number of propositions appearing on separate lines into a single proposition in a single step.

In the proof of 3.12.7, I avail myself of the first of these conventions at lines 9, 11, 13 and 15.

3.12.7 1. $(\exists x)\{Jx \bullet Hx \bullet (\forall y)[(Jy \bullet Hy) \supset x=y] \bullet Ex\}$ / $(\forall x)[(Jx \bullet Hx) \supset Ex]$

	2. $\sim(\forall x)[(Jx \bullet Hx) \supset Ex]$	AIP
	3. $(\exists x)\sim[(Jx \bullet Hx) \supset Ex]$	2, QE
	4. $\sim[(Ja \bullet Ha) \supset Ea]$	3, EI
	5. $\sim[\sim(Ja \bullet Ha) \vee Ea]$	4, Impl
	6. $\sim\sim(Ja \bullet Ha) \bullet \sim Ea$	5, DM
	7. $Ja \bullet Ha \bullet \sim Ea$	6, DN
	8. $Jb \bullet Hb \bullet (\forall y)[(Jy \bullet Hy) \supset b=y] \bullet Eb$	1, EI (to b)
	9. $(\forall y)[(Jy \bullet Hy) \supset b=y]$	8, Simp
	10. $(Ja \bullet Ha) \supset b=a$	9, UI (to a)
	11. $Ja \bullet Ha$	7, Simp
	12. $b=a$	10, 11, MP
	13. Eb	8, Simp
	14. Ea	13, 12, IDi
	15. $\sim Ea$	7, Simp
	16. $Ea \bullet \sim Ea$	14, 15, Conj
17. $(\forall x)[(Jx \bullet Hx) \supset Ex]$		2-16, IP

QED

3.12.8 contains another substantial proof. At 3.12.9, I show how you can shorten the proof using the rules of passage.

3.12.8 There is at least one moon of Earth.
There is at most one moon of Earth. / So, there is exactly one moon of Earth.

1. $(\exists x)Mx$
2. $(\forall x)(\forall y)[(Mx \bullet My) \supset x=y]$ / $(\exists x)[Mx \bullet (\forall y)(My \supset x=y)]$

3. Ma		1, EI
	4. $\sim(\exists x)[Mx \bullet (\forall y)(My \supset x=y)]$	AIP
	5. $(\forall x)\sim[Mx \bullet (\forall y)(My \supset x=y)]$	4, QE
	6. $(\forall x)[\sim Mx \vee \sim(\forall y)(My \supset x=y)]$	5, DM
	7. $\sim Ma \vee \sim(\forall y)(My \supset a=y)$	6, UI
	8. $\sim\sim Ma$	3, DN
	9. $\sim(\forall y)(My \supset a=y)$	7, 8, DS
	10. $(\exists y) \sim(My \supset a=y)$	9, QE
	11. $\sim(Mb \supset a=b)$	10, EI
	12. $\sim(\sim Mb \vee a=b)$	11, Impl
	13. $\sim\sim Mb \bullet \sim a=b$	12, DM
	14. $Mb \bullet \sim a=b$	13, DN
	15. Mb	14, Simp
	16. $\sim a=b$	14, Simp
	17. $(\forall y)[(Ma \bullet My) \supset a=y]$	2, UI
	18. $(Ma \bullet Mb) \supset a=b$	17, UI
	19. $Ma \bullet Mb$	3, 15, Conj
	20. $a=b$	18, 19, MP
	21. $a=b \bullet \sim a=b$	20, 16, Conj
22. $\sim\sim(\exists x)[Mx \bullet (\forall y)(My \supset x=y)]$		4-21, IP
23. $(\exists x)[Mx \bullet (\forall y)(My \supset x=y)]$		22, DN

QED

3.12.9	1. $(\exists x)Mx$	
	2. $(\forall x)(\forall y)[(Mx \bullet My) \supset x=y]$	/ $(\exists x)[Mx \bullet (\forall y)(My \supset x=y)]$
	3. $(\forall x)(\forall y)[Mx \supset (My \supset x=y)]$	1, Exp
	4. $(\forall x)[Mx \supset (\forall y)(My \supset x=y)]$	RP8
	5. Ma	1, EI
	6. $Ma \supset (\forall y)(My \supset a=y)]$	3, UI
	7. $(\forall y)(My \supset a=y)]$	6, 5, MP
	8. $Ma \bullet (\forall y)(My \supset a=y)]$	5, 7, Conj
	9. $(\exists x)[Mx \bullet (\forall y)(My \supset x=y)]$	8, EG

QED

3.12.10 has an even longer derivation. Notice that my use of our new conventions (especially at lines 27, 33, and 40) does not keep the proof short!

3.12.10 There are at least two cars in the driveway.
All the cars in the driveway belong to John.
John has at most two cars. / So, there are exactly two cars in the driveway.

1. $(\exists x)(\exists y)(Cx \bullet Dx \bullet Cy \bullet Dy \bullet x \neq y)$
2. $(\forall x)[(Cx \bullet Dx) \supset Bxj]$
3. $(\forall x)(\forall y)(\forall z)[(Cx \bullet Bxj \bullet Cy \bullet Byj \bullet Cz \bullet Bzj) \supset (x=y \vee x=z \vee y=z)]$
 / $(\exists x)(\exists y)\{Cx \bullet Dx \bullet Cy \bullet Dy \bullet x \neq y \bullet (\forall z)[(Cz \bullet Dz) \supset (z=x \vee z=y)]$
4. $(\exists y)(Ca \bullet Da \bullet Cy \bullet Dy \bullet a \neq y)$ 1, EI
5. $Ca \bullet Da \bullet Cb \bullet Db \bullet a \neq b$ 4, EI
6. $Ca \bullet Da$ 5, Simp
7. $(Ca \bullet Da) \supset Baj$ 2, UI
8. Baj 7, 6, MP
9. $Cb \bullet Db$ 5, Simp
10. $(Cb \bullet Db) \supset Bbj$ 2, UI
11. Bbj 10, 9, MP
12. $a \neq b$ 5, Simp
 13. $\sim(\forall z)[(Cz \bullet Dz) \supset (z=a \vee z=b)]$ AIP
 14. $(\exists z)\sim[(Cz \bullet Dz) \supset (z=a \vee z=b)]$ 13, QE
 15. $(\exists z)\sim[\sim(Cz \bullet Dz) \vee (z=a \vee z=b)]$ 14, Impl
 16. $(\exists z)[\sim\sim(Cz \bullet Dz) \bullet \sim(z=a \vee z=b)]$ 15, DM
 17. $(\exists z)[(Cz \bullet Dz) \bullet \sim(z=a \vee z=b)]$ 16, DN
 18. $Cc \bullet Dc \bullet \sim(c=a \vee c=b)$ 17, EI
 19. Ca 6, Simp
 20. $Ca \bullet Baj$ 19, 8 Conj
 21. Cb 9, Simp
 22. $Cb \bullet Bbj$ 21, 11, Conj
 23. $Cc \bullet Dc$ 18, Simp
 24. $(Cc \bullet Dc) \supset Bcj$ 2, UI
 25. Bcj 24, 23, MP
 26. Cc 23, Simp
 27. $Cc \bullet Bcj$ 26, 25, Conj
 28. $Ca \bullet Baj \bullet Cb \bullet Bbj \bullet Cc \bullet Bcj$ 20, 22, 27, Conj
 29. $(\forall y)(\forall z)[(Ca \bullet Baj \bullet Cy \bullet Byj \bullet Cz \bullet Bzj) \supset (a=y \vee x=z \vee y=z)]$ 3, UI
 30. $(\forall z)[(Ca \bullet Baj \bullet Cb \bullet Bbj \bullet Cz \bullet Bzj) \supset (a=b \vee a=z \vee b=z)]$ 29, UI
 31. $(Ca \bullet Baj \bullet Cb \bullet Bbj \bullet Cc \bullet Bcj) \supset (a=b \vee a=c \vee b=c)$ 30, UI
 32. $a=b \vee a=c \vee b=c$ 31, 28, MP
 33. $a \neq b$ 5, Simp
 34. $a=c \vee b=c$ 32, 33, DS
 35. $\sim(c=a \vee c=b)$ 18, Simp
 36. $\sim(c=a \vee b=c)$ 35, IDs
 37. $\sim(a=c \vee b=c)$ 36, IDs
 38. $(a=c \vee b=c) \bullet \sim(a=c \vee b=c)$ 34, 37, Conj
39. $\sim\sim(\forall z)\ (Cz \bullet Dz) \supset (z=a \vee z=b)]$ 13-38, IP
40. $(\forall z)[(Cz \bullet Dz) \supset (z=a \vee z=b)]$ 39, DN
41. $Ca \bullet Da \bullet Cb \bullet Db \bullet a \neq b \bullet (\forall z)[(Cz \bullet Dz) \supset (z=a \vee z=b)]$ 6, 9, 12, 40,Conj
42. $(\exists y)\{Ca \bullet Da \bullet Cy \bullet Dy \bullet a \neq y \bullet (\forall z)[(Cz \bullet Dz) \supset (z=a \vee z=y)]\}$ 41, EG
43. $(\exists x)(\exists y)\{Cx \bullet Dx \bullet Cy \bullet Dy \bullet x \neq y \bullet (\forall z)[(Cz \bullet Dz) \supset (z=x \vee z=y)]\}$ 42, EG

QED

Exercises 3.12a. Derive the conclusions of each of the following arguments.

1.
1. Dkm • (∀x)(Dkx ⊃ x=m)
2. Dab
3. Fb • ~Fm / a≠k

2.
1. (∃x)(∃y)[(Fx • Fy) ⊃ x=y]
2. (∀x)(∀y)x≠y / Fa ⊃ ~Fb

3.
1. (∀x)[(∃y)Pxy ⊃ (∃z)Pzx]
2. (∃x)(Pxb • x=d) / (∃z)Pzd

4.
1. (∀x)[Jx ∨ (Kx • Lx)]
2. ~(Ja ∨ Kb) / a≠b

5.
1. (∀x)[(Mx ∨ Nx) ⊃ Ox]
2. ~Oc
3. Md / c≠d

6.
1. (∀x)(Qx ⊃ Sx)
2. (∀x)(Rx ⊃ Tx)
3. (∀x)[Qx ∨ (Rx • Ux)]
4. a=b / Sb ∨ Ta

7.
1. Fac • Fbc • (∀x)[Fxc ⊃ (x=a ∨ x=b)]
2. (∃x)(Fxc • x≠a)
3. Fb • Gb / Fd • Gd

8.
1. (∀x)(∀y)[Ax ⊃ (By ⊃ Cxy)]
2. Aa • Ba
3. a=b / Cab

9.
1. (∃x)(Mx • Px)
2. (∀x)[Mx ⊃ (∀y)(Ky ⊃ x=y)]
3. Kf / Mf • Pf

10.
1. (∀x)[Ax ∨ (Bx • Cx)]
2. ~(∀x)Bx
3. (∀x)(Ax ⊃ x=c) / (∃x)x=c

11.
1. Dp • (∃x)(Ex • ~Fxp)
2. (∀x)[Gx ⊃ (∀y)Fyx] / (∃x)(Gx • ~Dx)

12.
1. Ha • Ia • (∀x)[(Hx • Ix) ⊃ x=a]
2. Hb • Jb • (∀x)[(Hx • Jx) ⊃ x=b]
3. Ka • ~Kb /~(∃x)(Hx • Ix • Jx)

13. 1. $(\exists x)(\exists y)(Lx \bullet Ly \bullet x \neq y)$
2. $(\forall x)(\forall y)(\forall z)[(Lx \bullet Ly \bullet Lz) \supset (x=y \vee y=z \vee x=z)]$
3. $La \bullet Lb$ $/ (\forall x)[Lx \supset (x=a \vee x=b)]$

14. 1. $(\forall x)(Ecx \supset x=d)$
2. $(\forall x)\{(Fx \bullet Gx) \supset (\forall y)[(Fy \bullet Gy) \supset y=x]\}$
3. $(\exists x)(Fx \bullet Gx \bullet Ecx)$
4. $Fa \bullet Ga$ $/ a=d$

15. 1. $(\exists x)(\exists y)(Hx \bullet Ix \bullet Jx \bullet Hy \bullet Iy \bullet Jy \bullet x \neq y)$
2. $(\forall x)(\forall y)[(Hx \bullet Ix \bullet Jx \bullet Hy \bullet Iy \bullet Jy) \supset x=y]$
$/ (\exists x)(\exists y)\{Hx \bullet Ix \bullet Jx \bullet Hy \bullet Iy \bullet Jy \bullet x \neq y \bullet (\forall z)[(Hz \bullet Iz \bullet Jz) \supset (z=x \vee z=y)]\}$

16. 1. $Na \bullet Oa \bullet Nb \bullet Ob \bullet a \neq b \bullet (\forall x)[(Nx \bullet Ox) \supset (x=a \vee x=b)]$
2. $Na \bullet {\sim}Pa \bullet (\forall x)[(Nx \bullet x \neq a) \supset Px]$ $/ (\exists x)\{Nx \bullet Ox \bullet Px \bullet (\forall y)[(Ny \bullet Oy \bullet Py) \supset y=x]\}$

17. 1. $(\exists x)(\exists y)(Kx \bullet Lx \bullet Ky \bullet Ly \bullet x \neq y)$
2. $Ka \bullet La \bullet Ma \bullet (\forall y)[(Ky \bullet Ly \bullet My) \supset y=a]$
$/ (\exists x)(Kx \bullet Lx \bullet {\sim}Mx)$

18. 1. $(\exists x)(\exists y)(Ax \bullet Cx \bullet Ay \bullet Cy \bullet x \neq y)$
2. $(\forall x)(\forall y)(\forall z)[(Cx \bullet Cy \bullet Cz) \supset (x=y \vee x=z \vee y=z)]$
3. $(\exists x)(Bx \bullet {\sim}Ax)$ $/ {\sim}(\forall x)(Bx \supset Cx)$

19. 1. $(\exists x)(\exists y)(Qx \bullet Rx \bullet Qy \bullet Ry \bullet x \neq y)$
2. $(\forall x)(\forall y)(\forall z)[(Rx \bullet Sx \bullet Ry \bullet Sy \bullet Rz \bullet Sz) \supset (x=y \vee x=z \vee y=z)]$
3. $(\forall x)({\sim}Qx \vee Sx)$
$/ (\exists x)(\exists y)\{Qx \bullet Rx \bullet Sx \bullet Qy \bullet Ry \bullet Sy \bullet x \neq y \bullet (\forall z)[(Rz \bullet Sz) \supset (z=x \vee z=y)]\}$

20. 1. $Ma \bullet {\sim}Pa \bullet Mb \bullet {\sim}Pb \bullet (\forall x)[(Mx \bullet x \neq a \bullet x \neq b) \supset Px]$
2. $Qb \bullet (\forall x)[(Mx \bullet Qx) \supset x=b]$
3. $(\forall x)\{Mx \supset [{\sim}(Qx \vee Px) \equiv Rx]\}$
4. $a \neq b$ $/ (\exists x)\{Mx \bullet Rx \bullet (\forall y)[(My \bullet Ry) \supset y=x]\}$

Exercises 3.12b. Translate and derive.

1. Polly flies. Olivia doesn't. So, Polly is not Olivia. (o, p, Fx)

2. If George is Dr. Martin, then Dr. Martin is married to Mrs. Wilson. Dr. Martin is George. Mrs. Wilson is Hilda. So, George is married to Hilda. (g, h, m, w, Mxy: x is married to y)

3. If something is a not superhero, then everything is not Wonder Woman. So, Wonder Woman is a superhero. (w, Sx)

4. Katerina is the fastest runner on the team. Pedro is a runner on the team. Katerina is not Pedro. So, Katerina is faster than Pedro. (k, p, Rx, Tx, Fxy: x is faster than y)

5. Ryan is the only professor who teaches Metaphysics at the college. All engaging professors win an award. All professors at the college are engaging. So, Ryan wins an award. (m, r, Cx, Ex, Px, Wx, Txy: x teaches y)

6. The author of Republic was a Greek philosopher. John Locke was a philosopher, but he was not Greek. Therefore, John Locke did not write Republic. (l, r, Gx, Px, Wxy: x wrote y)

7. The only person who went skiing was James. The only person who caught a cold was Mr. Brown. Some person who went skiing also caught a cold. So, James is Mr. Brown. (j, b, Cx, Px, Sx)

8. Every student except Paco writes a thesis. Every student except Ricardo gives a presentation. Paco is not Ricardo. So, Paco gives a presentation and Ricardo writes a thesis. (p, r, Gx, Sx, Wx)

9. Exactly one student in the class gives a presentation about Spinoza. At least two students in the class give a presentation about Leibniz. No student in the class gives a presentation about both Leibniz and Spinoza. So, there are at least three students in the class. (l, s, Sx, Gxy: x gives a presentation about y)

10. Every employee except Rupert got a promotion. The only employee to get a promotion was Jane. So, there are exactly two employees. (Ex: x is an employee, r: Rupert, Px: x gets a promotion, j: Jane)

§3.13: Functions

In the last two sections of this chapter, we will look at two final, formal topics: functions and higher-order quantification. These two extensions of logic are contentious, and many philosophers do not consider them to be logical at all. But, their introduction facilitates inferences which appear to be logical.

Consider, as a motivating example to introduce functions, the intuitively-valid argument 3.13.1.

3.13.1 All applicants will get a job.
Jean is an applicant.
Jean is the first child of Dominique and Henri.
So, some first child will get a job.

The first two premises are easily regimented into **F**.

3.13.2 1. $(\forall x)(Ax \supset Gx)$
2. Aj

We have several options for the third premise. We could take 'first child of Dominique and Henri' as a monadic predicate, as at 3.13.3.

3.13.3 3. Fj

Then we would need a different predicate for being the first child (of any couple) for the conclusion. Alternatively, we could regiment the third premise by using Russell's theory of definite descriptions, using 'Fxyz' for 'x is a first child of y and z', and adding a uniqueness clause.

3.13.4 3. $(\exists x)[Fxdh \bullet (\forall y)(Fydh \supset y=x) \bullet x=j]$

3.13.4 has the advantage of taking 'first child of' to be a three-place relation. That option reveals more logical structure than 3.13.3, and so may be useful. Correspondingly, we can regiment the conclusion of 3.13.1 as 3.13.5.

3.13.5 $(\exists x)\{(\exists y)(\exists z)[Fxyz \bullet (\forall w)(Fwyz \supset w=x)] \bullet Gx\}$

The conclusion 3.13.5 follows from the premises at 3.13.2 and 3.13.4, as we can see at 3.13.6.

3.13.6	1. $(\forall x)(Ax \supset Gx)$	
	2. Aj	
	3. $(\exists x)[Fxdh \bullet (\forall y)(Fydh \supset y=x) \bullet x=j]$	
	/ $(\exists x)\{(\exists y)(\exists z)[Fxyz \bullet (\forall w)(Fwyz \supset w=x)] \bullet Gx\}$	
	4. $Fadh \bullet (\forall y)(Fydh \supset y=a) \bullet a=j$	3, EI
	5. $a=j$	4, Simp
	6. $j=a$	5, IDs
	7. Aa	2, 6, IDi
	8. $Aa \supset Ga$	1, UI
	9. Ga	8, 7, MP
	10. $(\forall y)(Fydh \supset y=a)$	4, Simp
	11. $Fwdh \supset w=a$	10, UI
	12. $(\forall w)(Fwdh \supset w=a)$	11, UG
	13. $Fadh$	4, Simp
	14. $Fadh \bullet (\forall w)(Fwdh \supset w=a)$	13, 12, Conj
	15. $(\exists z)[Fadz \bullet (\forall w)(Fwdz \supset w=a)]$	14, EG
	16. $(\exists y)(\exists z)[Fayz \bullet (\forall w)(Fwyz \supset w=a)]$	15, EG
	17. $(\exists y)(\exists z)[Fayz \bullet (\forall w)(Fwyz \supset w=a)] \bullet Ga$	16, 9, Conj
	18. $(\exists x)\{(\exists y)(\exists z)[Fxyz \bullet (\forall w)(Fwyz \supset w=x)] \bullet Gx\}$	17, EG

QED

The derivation at 3.13.6 is successful, but there is a more efficient, and more fecund, option for regimenting ‘the first child of x and y’. We can take ‘the first child of x and y’ to be a function. This option will allow us to regiment both the third premise and the conclusion more simply. It will also allow us to construct simpler derivations.

We have seen that we can, using the identity predicate, simulate adjectival uses of numbers. With a small extension of **F**, adding functors like ‘f(x)’, we can express even more mathematics. A functor is a symbol used to represent a function, like any of the functions ubiquitous in and essential for mathematics and science. In mathematics, there are linear function, exponential functions, periodic functions, quadratic functions, and trigonometric functions. In science, force is a function of mass and acceleration, momentum is a function of mass and velocity, even genetic code is a function.

The utility of functions to mathematics makes them suspect as logic. But understanding functions is essential for work in metalogic. Recall that the semantics for **PL** is presented in terms of truth functions. All the connectives are truth functions, taking one argument (negation) or two arguments (the rest of the connectives) and yielding a specific truth value.

Consider terms like ‘the father of’, ‘the successor of’, ‘the sum of’, and ‘the adviser of’. Each takes one or more arguments, from their domain, and produces a single output, the range. (A student might have more than one adviser, but let’s imagine not, for the moment.) One-place functions take one argument, two-place functions take two arguments, n-place functions take n arguments.

3.13.7 lists some functions and their logical representations.

3.13.7	the father of:	$f(x)$
	the successor of:	$g(x)$
	the sum of:	$f(x,y)$
	the teacher of:	$g(x_1 \ldots x_n)$

The last function can take as arguments, say, all the students in a class.

An essential characteristic of functions is that they yield exactly one value, no matter how many arguments they take. Thus, the expressions at 3.13.8 are not functions.

3.13.8 the parents of a
the classes that a and b share
the square root of x

These expressions are relations; a function is a special type of relation. By adding functors to our language **F**, we have adopted a new language, which I call **FF**, for Full First-Order Predicate logic with functors.

Vocabulary of FF

Capital letters A...Z, used as predicates
Lower case letters
a, b, c, d, e, i, j, k...u are used as constants.
f, g, and h are used as functors.
v, w, x, y, z are used as variables.
Five connectives: ~, •, ∨, ⊃ ≡
Quantifiers: ∃, ∀
Punctuation: (), [], {}

In order to specify the formation rules for **FF**, we have to invoke n-tuples of singular terms. An **n-tuple of singular terms** is an ordered series of singular terms (constants, variables, or functor terms). N-tuples differ from sets in that the order of their arguments matters. 'N-tuple' is a general term that covers 'single', 'double', 'triple', 'quadruple', etc. We use that term since functions can take any number of arguments. Often, an n-tuple is represented thus: <a, b, c>. We will represent n-tuples merely by listing the singular terms separated by commas, as at 3.13.9.

3.13.9	a,b	two arguments
	a,a,f(a)	three arguments
	x,y,b,d,f(x),f(a,b,f(x))	six arguments
	a	one argument

Suppose α is an n-tuple of singular terms. Then the expressions at 3.13.10 are all **functor terms**.

3.13.10 f(α)
g(α)
h(α)

Note that an n-tuple of singular terms can include functor terms. 'Functor term' is defined recursively, which allows for composition of functions. For example, one can refer to the grandfather of x, using just the functions for father, e.g. f(x), and mother, e.g. g(x).

3.13.11 f(f(x))
3.13.12 f(g(x))

3.13.11 represents 'paternal grandfather' and 3.13.12 represents 'maternal grandfather'. Similarly, if we take 'h(x)' to represent the square of x, 3.13.13 represents the eighth power of x, i.e. $((x^2)^2)^2$.

3.13.13 h(h(h(x)))

I have introduced only three functor letters. As with variables and constants, there are several different tricks for constructing an indefinite number of terms out of a finite vocabulary, using indexing. We won't need more than the three letters. Even with just the three letters, we have an indefinite number of functors, since each of 3.13.14 is technically a different functor, and can represent a different function.

3.13.14 f(a)
f(a,b)
f(a,b,c)
f(a,b,c,d)
etc.

The scope and binding rules are the same for **FF** as they were for **M** and **F**. The formation rules only need one small adjustment, at the first line.

Formation rules for wffs of FF.

1. An n-place predicate followed by n singular terms (constants, variables, **or functor term**s) is a wff.
2. For any variable β, if α is a wff that does not contain either '(∃β)' or '(∀β)', then '(∃β)α' and '(∀β)α' are wffs.
3. If α is a wff, so is ~α.
4. If α and β are wffs, then so are:
   (α • β)
   (α ∨ β)
   (α ⊃ β)
   (α ≡ β)
5. These are the only ways to make wffs.

The semantics for **FF** are basically the same as for **F**. For an interpretation of **FF**, we insert an interpretation of function symbols.

Specifying a semantics for FF

Step 1. Specify a set to serve as a domain of interpretation, or domain of quantification.
Step 2. Assign a member of the domain to each constant.
Step 3. Assign a function with arguments and ranges in the domain to each function symbol.
Step 4. Assign some set of objects in the domain to each one-place predicate; assign sets of ordered n-tuples to each relational predicate
Step 5. Use the customary truth tables for the interpretation of the connectives.

The function assigned in Step 3 will be a function in the metalanguage used to interpret the function in the object language. I won't pursue a discussion of metalinguistic functions, except to say that they work just like ordinary mathematical functions. Once you have the idea of how functions work in the object language, it will become clear how they work in the metalanguage.

Translations into FF and simple arithmetic functions

At 3.13.15, there is a translation key and some sentences of **FF**.

3.13.15

Lxy: x loves y
f(x): the father of x
g(x): the mother of x
o: Olaf

Olaf loves his mother.	Log(o)
Olaf loves his grandmothers.	Log(g(o)) • Log(f(o))
Noone is his/her own mother.	(∀x)~x=g(x)

Many simple concepts in arithmetic are functions: addition, multiplication, least common multiple. The most fundamental function in mathematics is the successor function. All other mathematical functions can be defined in terms of successor and other basic concepts. In fact, all of arithmetic can be developed from five basic axioms, called the Peano axioms. They are named for Giuseppe Peano, who published in 1889 a precise version of the axioms that Richard Dedekind had published a year earlier. Peano had credited Dedekind, and sometimes these axioms are called the Dedekind-Peano, or even the Dedekind, axioms.

3.13.16 Peano's Axioms for Arithmetic[4]

P1: 0 is a number
P2: The successor (x') of every number (x) is a number
P3: 0 is not the successor of any number
P4: If x'=y' then x=y
P5: If P is a property that may (or may not) hold for any number, and if
  i. 0 has P; and
  ii. for any x, if x has P then x' has P;
  then all numbers have P.

P5 is called the induction schema, and is actually a schema of an infinite number of axioms. Mathematical induction is essential in advanced logic, as well as in linear algebra and number theory. We can write the Peano axioms in **FF** if we use the given key.

a: zero
Nx: x is a number
f(x): the successor of x

P1. Na
P2. (∀x)(Nx ⊃ Nf(x))
P3. ~(∃x)(Nx • f(x)=a)
P4. (∀x)(∀y)[(Nx • Ny) ⊃ (f(x)=f(y) ⊃ x=y)]
P5. {Pa • (∀x)[(Nx • Px) ⊃ Pf(x)]} ⊃ (∀x)(Nx ⊃ Px)

[4] Following Mendelson 1997.

3.13.17 - 3.13.19 present a few translations of arithmetic sentences using functions. Note that in the following sentences, I take 'number' to mean 'natural number', and use the following translation key.

o: one	Ex: x is even
f(x): the successor of x	Ox: x is odd
f(x, y): the product of x and y	Px: x is prime

3.13.17 One is not the successor of any number.

$$(\forall x)(Nx \supset {\sim}f(x){=}o)$$

3.13.18 If the product of a pair of numbers is odd, then the product of the successors of those numbers is even.

$$(\forall x)(\forall y)\{(Nx \bullet Ny) \supset [Of(x, y) \supset Ef(f(x), f(y))]\}$$

3.13.19 There are no prime numbers such that their product is prime.

$$\sim(\exists x)(\exists y)[Nx \bullet Px \bullet Ny \bullet Py \bullet Pf(x, y)]$$

Derivations using functions

There are no new rules covering functions, which act like simple singular terms. Since a function always stands for a single element from the domain, no matter how many arguments it takes, we can consider a function as if it were either a constant or a variable. Whether we should treat a function as a constant or a variable for the purposes of instantiating or generalizing depends both on the arguments of the function and the particular rule. As always, UI and EG are free of restrictions. Since a universal claim is true of everything, it is true of both simple singular terms and complex singular terms. We can UI to a variable, or to a function of a variable, or to any complex function all of whose arguments are variables.

The inferences at 3.13.20 are all acceptable using UI.

3.13.20 $(\forall x)(Px \supset Qx)$

$Pa \supset Qa$

$Px \supset Qx$

$Pf(x) \supset Qf(x)$

$Pf(a) \supset Qf(a)$

$Pf(f(g(f(a)))) \supset Qf(f(g(f(a))))$

$Pf(f(g(f(x)))) \supset Qf(f(g(f(x))))$

Similarly, since existentially quantified sentences are so weak, merely claiming that some object in the domain has a property, we can EG at any point over any singular terms. '$(\exists x)(Px \bullet Qx)$' can be inferred from any of the statements listed at 3.13.21.

3.13.21 $Pa \bullet Qa$

$Pf(a) \bullet Qf(a)$

$Pf(x) \bullet Qf(x)$

$Pf(f(g(f(a)))) \bullet Qf(f(g(f(a))))$

$Pf(f(g(f(x)))) \bullet Qf(f(g(f(x))))$

Moreover, all of the inferences at 3.13.22 are acceptable using EG.

3.13.22 $\quad$ Pf(f(g(f(a)))) • Qf(f(g(f(a))))

(∃x)[Pf(f(g(f(a)))) • Qf(f(g(f(a))))]
(∃x)[Pf(f(g(x))) • Qf(f(g(x)))]
(∃x)[Pf(f(x)) • Qf(f(x))]
(∃x)[Pf(x) • Qf(x)]

The examples at 3.13.20 - 3.13.22 extend naturally to functions of more than one argument.

Using UG with functions requires some care. Consider the faulty derivation at 3.13.23.

3.13.23	1. (∀x)Pf(x)	Premise
	2. Pf(x)	1, UI
	3. (∀x)Px	2, UG: but wrong!

Uh-oh!

The problem with 3.13.23 is clear if we interpret 'Px' as 'x is even' and 'f(x)' as the doubling function. Then, we have concluded that all numbers are even from the premise that all doubles are even.

Using EI with functions also requires care. The following inference 3.13.24 is also fallacious.

3.13.24	1. (∃x)Px	Premise
	2. Pf(a)	1, EI: but wrong!

Uh-oh!

Again, we can see the problem with 3.13.24 by interpreting the symbols of the inference. Let's interpret 'Px' as 'x is odd' and 'f(x)' as the doubling function. Then, 3.13.24 concludes that some double is odd from the premise that some number is odd.

The solution to both faulty inferences 3.13.23 and 3.13.24 involves making sure that when you use EI and UG, you leave the functors as they were. Do not eliminate functors when using UG. Do not introduce functors when using EI.

For UG, if the arguments of a function are all variables, then you are free to use UG over the variables in that function, though the earlier restrictions on UG continue to hold. If the arguments of a function contain any constants, then you may not use UG.

For EI, we must continue always to instantiate to a new singular term. A functor is not a new singular term if any of its arguments, or any of the arguments of any of its sub-functors, have already appeared in the derivation or appear in the conclusion.

Let's return to argument 3.13.1. We saw at 3.13.6 that the conclusion follows if we regiment the argument as the form 3.13.25.

3.13.25 $\quad$ 1. (∀x)(Ax ⊃ Gx)
2. Aj
3. (∃x)[Fxdh • (∀y)(Fydh ⊃ y=x) • x=j]
/ (∃x){(∃y)(∃z)[Fxyz • (∀w)(Fwyz ⊃ w=x)] • Gx}

I also mentioned that invoking functions would make the derivation simpler. Let's use a function 'f(x,y)' for 'the first child of x and y' to regiment the third premise and conclusion. The derivation follows quickly.

3.13.26	1. $(\forall x)(Ax \supset Gx)$		
	2. Aj		
	3. $j=f(d,h)$	/ $(\exists x)(\exists y)Gf(x,y)$	
	4. $Aj \supset Gj$	1, UI	
	5. Gj	4, 2, MP	
	6. $Gf(d,h)$	5, 3, IDi	
	7. $(\exists y)Gf(d,y)$	6, EG	
	8. $(\exists x)(\exists y)Gf(x,y)$	7, Eg	

QED

3.13.27 contains a derivation which uses some composition of functions. Note that ‘B’ is a two-place predicate, taking as arguments a variable and a functor term with a variable argument in the first premise, and taking as arguments two functor terms, each with variable arguments, in the conclusion.

3.13.27	1. $(\forall x)[Ax \supset Bf(x)x]$		
	2. $(\exists x)Af(x)$	/ $(\exists x)Bf(x)f(f(x))$	
	3. $Af(a)$	2, EI	to ‘a’
	4. $Af(a) \supset Bf(a)f(f(a))$	1, UI	to ‘f(a)’
	5. $Bf(a)f(f(a))$	4, 3, MP	
	6. $(\exists x)Bf(x)f(f(x))$	5, EG	

QED

In the short derivation 3.13.28, we instantiate to a two-place function, f(g(x), x), one of whose arguments is itself a function. Since none of the arguments of any of the functions in 3.13.28 are constants, UG is permissible at line 3.

3.13.28	1. $(\forall x)Cx$	/ $(\forall x)Cf(f(x), x)$
	2. $\sim Cf(x, g(x))$	1, UI
	3. $(\forall x)\sim Cf(x, g(x))$	2, UG

QED

3.13.29 derives the conclusion of an argument which uses concepts from number theory in which functions play an important role.

3.13.29
1. If the product of a pair of numbers is odd, then the product of the successors of those numbers is even.
2. Seven and three are odd numbers.
3. The product of seven and three is odd.
So, the product of the successors of seven and three is even.

1. $(\forall x)(\forall y)\{(Nx \bullet Ny) \supset [Of(x, y) \supset Ef(f(x), f(y))]\}$		
2. $Os \bullet Ns \bullet Ot \bullet Nt$		
3. $Of(s, t)$	/ $Ef(f(s), f(t))$	
4. $(\forall y)\{(Ns \bullet Ny) \supset [Of(s, y) \supset Ef(f(s), f(y))]\}$		1, UI
5. $(Ns \bullet Nt) \supset [Of(s, t) \supset Ef(f(s), f(t))]$		4, UI
6. $Ns \bullet Nt$		2, Simp
7. $Of(s, t) \supset Ef(f(s), f(t))$		5, 6, MP
8. $Ef(f(s), f(t))$		7, 3, MP

QED

Exercises 3.13a. Use the given key to translate the following sentences into **FF**.

For exercises 1-8 use the following key:

m: Mariel
Sxy: x is a sister of y
Txy: x takes care of y
Px: x is a person
f(x): the mother of x
g(x): the father of x

1. Mariel takes care of her mother.
2. Mariel's paternal grandmother takes care of Mariel.
3. Mariel takes care of her grandmothers.
4. Mariel's sister takes care of Mariel's grandfathers.
5. Mariel's only sister takes care of Mariel's grandfathers.
6. No one is his/her own mother.
7. Not everyone is the father of someone.
8. Some maternal grandmothers are sisters to someone.

For exercises 9-16 use the following key:

t: two
Ex: x is even
Ox: x is odd
Nx: x is a number
Px: x is prime
f(x): the square of x
g(x): the successor of x
f(x, y): the product of x and y

9. Two and its successor are prime numbers.
10. Not all odd numbers are prime.
11. The square of an odd number is odd.
12. The square of a number is not prime.
13. The product of even numbers is even.
14. The product of a number and its successor is not prime.
15. The product of an odd number and an even number is even.
16. The square of a number is the product of it with itself.

Exercises 3.13b. Derive the conclusions of each of the following arguments.

1.
 1. $(\forall x)(Ax \supset Af(x))$
 2. Aa
 3. $f(a)=b$ / Ab

2.
 1. $(\forall x)[Bx \equiv Bg(x)]$
 2. $(\forall x)g(x)=f(x,x)$
 3. Ba / $Bf(a,a)$

3.
 1. $(\forall x)Hf(x)$
 2. $a=f(b) \bullet b=f(c)$
 3. $(\forall x)(Hx \supset {\sim}Ix)$ / $a=f(f(c)) \bullet {\sim}Ia$

4.
 1. $(\forall x)[(Bf(x) \supset (Cx \bullet Df(f(x)))]$
 2. $(\exists x)Bf(f(x))$
 3. $(\exists x)Cf(x) \supset (\forall x)Ex$ / $(\exists x)[Df(f(f(x))) \bullet Ef(f(f(x)))]$

5.
 1. $(\forall x)(\forall y)[(Fx \bullet Fy) \supset Gf(x,y)]$
 2. $(\forall x)(\forall y)[Gf(x,y) \equiv Gf(x,x)]$
 3. $Fa \bullet Fb$ / $Gf(f(a,a))$

6.
 1. $f(a,b,c)=d$
 2. $(\forall x)(\forall y)(\forall z)(\forall w)\{f(x,y,z)=w \supset [Jw \vee Jf(w)]\}$
 3. $(\forall x)(Jx \supset Kx)$ / $Kd \vee Kf(d)$

7.
 1. $(\forall x)[(Px \bullet Qx) \supset Rf(x)]$
 2. $(\forall x)[Rx \supset (\exists y)Pxy]$
 3. ${\sim}(\forall x)(Px \supset {\sim}Qx)$ / $(\exists x)(\exists y)Pxy$

8.
 1. $(\forall x)(\forall y)[(Pxy \bullet Qxy) \supset {\sim}f(x)=y]$
 2. $(\forall x)(\forall y)[Qxy \equiv Qxf(y)]$
 3. $f(a)=b \bullet f(b)=a$
 4. Pab / ${\sim}Qaa$

9.
 1. $(\forall x)(\forall y)\{Qf(x,y) \supset [(Px \bullet Qy) \vee (Py \bullet Qx)]\}$
 2. $(\forall x)[Px \supset Qf(x)]$
 3. $(\forall x)Qf(x,f(x))$
 4. ${\sim}Pa$ / $Qa \bullet Pf(a)$

10.
 1. $(\forall x)(\forall y)\{Pf(x,y) \supset [(Px \bullet Py) \vee (Qx \bullet Qy)]\}$
 2. $(\forall x)[Px \supset Pf(f(x)]$
 3. $(\forall x)Pf(x, f(f(x)))$
 4. $(\exists x){\sim}Qx$ / $(\exists x)Pf(f(a))$

§3.14: Higher-Order Quantification

Our last logical language is a controversial extension of predicate logic. As always, we start with an inference that seems intuitively valid and look for a way to express the inference in our formal logic.

3.14.1 There are red apples.
There are red fire trucks.
So, some apples and some fire trucks have something in common.

A natural way to express the inference at 3.14.1 is to quantify over the predicates themselves. In other words, we can treat the predicates as if they are variables and quantify over them as we do in line 8 of 3.14.2.

3.14.2

	1. $(\exists x)(Rx \bullet Ax)$	
	2. $(\exists x)(Rx \bullet Fx)$	
	3. $Ra \bullet Aa$	1, EI
	4. $Rb \bullet Ab$	3, EI
	5. Ra	3, Simp
	6. Rb	4, Simp
	7. $Ra \bullet Rb$	5, 6, Conj
	8. $(\exists X)(Xa \bullet Xb)$	7, by existential generalization over predicates

A language which allows quantification over predicate places is called a second-order language. A system of logic which uses a second-order language is called second-order logic. I'll proceed to introduce a second-order language, which I'll call **S**.

We have previously allowed all capital letters to be predicate constants. In our new second-order logic, we are going to reserve 'V', 'W', 'X', 'Y', and 'Z' as predicate variables. Introducing predicate variables allows us to regiment some new sentences.

3.14.3 No two distinct things have all properties in common.
$(\forall x)(\forall y)[x \neq y \supset (\exists X)(Xx \bullet {\sim}Xy)]$

3.14.4 Identical objects share all properties.
$(\forall x)(\forall y)[x = y \supset (\forall Y)(Yx \equiv Yy)]$

3.14.4 is Leibniz's law. We saw Leibniz's law and its converse, the identity of indiscernibles, at 3.11.8 and 3.11.9, written as schematic sentences in the metalanguage. In a second-order language, we can write them as simple object-level sentences. The identity of indiscernibles is 3.14.5.

3.14.5 $(\forall x)(\forall y)[(\forall Z)(Zx \equiv Zy) \supset x = y]$

The law of the excluded middle, which we saw as a metalinguistic schematic sentence at 1.6.5, is also neatly regimented in second-order logic, with sentential variables, which you may recall we can take as zero-place predicates.

3.14.6 $(\forall X)(X \vee {\sim}X)$

Second-order logic allows us to regiment analogies, like 3.14.7.

3.14.7 Cat is to meow as dog is to bark. $(\exists X)(Xcm \bullet Xdb)$

Actually, 3.14.7 contains a deviant use of constants. But, it provides an example of the power of the second-order quantifiers.

Recall the induction schema in Peano Arithmetic, which we saw as the fifth axiom at 3.13.16. Since it is a schema, the theory is not finitely axiomatizable: there are infinitely many instances of the schema. Second-order logic allows us to replace the induction schema with single axiom. 3.14.8 uses 'a' to stand for zero, 'Nx' for 'x is a number', 'f(x)' for the successor function, and quantifies over any mathematical property using 'X'.

3.14.8 $(\forall X)\{\{Na \bullet Xa \bullet (\forall x)[(Nx \bullet Xx) \supset Xf(x)]\} \supset (\forall x)(Nx \supset Xx)\}$

Vocabulary of S

Capital letters
- A...U, used as predicates
- V, W, X, Y, and Z, used as predicate variables

Lower case letters
- a, b, c, d, e, i, j, k...u are used as constants.
- f, g, and h are used as functors.
- v, w, x, y, z are used as singular variables.

Five connectives: $\sim$, $\bullet$, $\vee$, $\supset$ $\equiv$
Quantifiers: $\exists$, $\forall$
Punctuation: (), [], {}

Formation rules for wffs of S.

1. An n-place predicate or predicate variable followed by n singular terms (constants, variables, or functor terms) is a wff.
2. For any singular variable β, if α is a wff that does not contain either '$(\exists\beta)$' or '$(\forall\beta)$', then '$(\exists\beta)\alpha$' and '$(\forall\beta)\alpha$' are wffs.
3. For any predicate variable β, if α is a wff that does not contain either '$(\exists\beta)$' or '$(\forall\beta)$', then '$(\exists\beta)\alpha$' and '$(\forall\beta)\alpha$' are wffs.
4. If α is a wff, so is $\sim\alpha$.
5. If α and β are wffs, then so are:
   $(\alpha \bullet \beta)$
   $(\alpha \vee \beta)$
   $(\alpha \supset \beta)$
   $(\alpha \equiv \beta)$
6. These are the only ways to make wffs.

Let's return to the expressive powers of **S**. 3.14.9 - 3.14.11 show three simple translations.

3.14.9 Everything has some relation to itself. $(\forall x)(\exists V)Vxx$

3.14.10 All people have some property in common.
$(\forall x)(\forall y)[(Px \bullet Py) \supset (\exists Y)(Yx \bullet Yy)]$

3.14.11 No two people have every property in common.
$(\forall x)(\forall y)[(Px \bullet Py) \supset (\exists Z)(Zx \bullet \sim Zy)$

Second-order logic allows us to regiment three important characteristics of relations: reflexivity, symmetry, and transitivity. A relation is reflexive if every object bears that relation to itself. Being the same size as something is a reflexive relation. So is being equidistant from a given point. A relation is symmetric if whenever one thing bears that relation to another, the reverse is also true. Being a sibling is a symmetric relation. Being older than is asymmetric. Lastly, transitivity is exemplified by hypothetical syllogism. Being older than, or larger than, or earlier than are all transitive relations.

These three properties of relations are especially important because they characterize identity. We call any relation which is reflexive, symmetric, and transitive an equivalence relation. Identity is an equivalence relation.

Given any particular relation, call it 'R', we can express that the relation is reflexive, symmetric, or transitive without any use of second-order quantification, as I do at 3.14.12 - 3.14.14.

3.14.12 Reflexivity $(\forall x)Rxx$
3.14.13 Symmetry $(\forall x)(\forall y)(Rxy \equiv Ryx)$
3.14.14 Transitivity $(\forall x)(\forall y)(\forall z)[(Rxy \bullet Ryz) \supset Rxz]$

In mathematics, many relations, like 'greater than', are antisymmetric, which we can also represent.

3.14.15 Antisymmetry $(\forall x)(\forall y)(Rxy \equiv {\sim}Rxy)$

Second-order logic allows us to do more with these characteristics. We can quantify over them and make assertions concerning these properties.

3.14.16 Some relations are transitive. $(\exists X)(\forall x)(\forall y)(\forall z)[(Xxy \bullet Xyz) \supset Xxz]$

3.14.17 Some relations are symmetric, while some are antisymmetric.
$(\exists X)(\forall x)(\forall y)(Xxy \equiv Xyx) \bullet (\exists X)(\forall x)(\forall y)(Xxy \equiv {\sim}Xyx)$

The additional power of second-order logic also entails that we need not reserve a special identity predicate. Instead, we can just introduce it as shorthand for the second-order claim 3.14.17.

3.14.17 $x{=}y \rightleftarrows (\forall X)(Xx \equiv Xy)$

Higher-Order Logics

Second-order logic is only one of the higher-order logics. All logics beyond first-order logic are called higher-order logic. To create third-order logic, we introduce attributes of attributes, for which I will use boldfaced italics. 3.14.18 and 3.14.19 use third-order constants.

3.14.18 All useful properties are desirable. $(\forall X)(\boldsymbol{U}X \supset \boldsymbol{D}X)$

3.14.19 A man who possesses all virtues is a virtuous man, but there are virtuous men who do not possess all virtues.
$(\forall x)\{[Mx \bullet (\forall X)(\boldsymbol{V}X \supset Xx)] \supset Vx\} \bullet (\exists x)[Mx \bullet Vx \bullet (\exists X)(\boldsymbol{V}X \bullet {\sim}Xx)]$

Though we haven't introduced formation rules for third-order logic, and so we can't say that they are ill-formed, 3.14.18 and 3.14.19 are missing objects (terms) after some of their predicates. Still, they are clear enough to give us the idea of third-order variables, and proper regimentations would obscure the key idea. We won't spend much time on higher-order logics, so I won't trouble to get these right. The only higher-order concept that will be useful to us is a version of identity for properties, at 3.14.20.

3.14.20 There are at least two distinct properties.

One potential regimentation of 3.14.20 is 3.14.21.

3.14.21 $(\exists X)(\exists Y)X \neq Y$

But 3.14.21 is ill-formed, since we have not defined identity for predicates, and since there are no objects attached to the predicates. We do have the '$\equiv$', which is an equivalence relation among predicates. Thus, we can translate 3.14.20 as 3.14.22.

3.14.22 $(\exists X)(\exists Y)(\exists x){\sim}(Xx \equiv Yx)$

Of course, 3.14.22 only indicates that there are distinct monadic properties. In order to generalize these claims, higher-order logics are required.

We will not consider derivations in higher-order logics.

Exercises 3.14. Translate each of the following sentences into **S**. Exercises 16-20 are adapted from Spinoza's *Ethics*. The parenthetical citations are standard; for example, '1p2' refers to the second proposition in part 1.

1. Liza has some attributes, but she lacks some attributes.
2. Cristóbal and Dante share no properties.
3. Reva has at least two different properties.
4. Everyone shares some property with Tudor.
5. Everyone shares some property with some monkeys.
6. Some chemists share some property with Einstein.
7. Gillian shares some attributes with a famous scientist.
8. All psychologists and biologists have some property in common.
9. Alec shares some of his mother's properties. (f(x): the mother of x)
10. Ron has all of his father's properties. (g(x): the father of x)
11. Some attributes are properties of nothing.
12. Some relations are transitive.
13. Something lacks all symmetric relations.
14. Some relations are both reflexive and symmetric.
15. Something lacks all transitive relations.
16. Two substances, whose attributes are different, have no properties in common (1p2).
17. Two or more distinct things are distinguished by the difference of their attributes (1p4).
18. There cannot exist in the universe two or more substances having some attribute (1p5).
19. Two things with no common properties cannot be the cause of one another (1p3). (f(x): the cause of x)
20. Two things with no common properties cannot be understood through each other (1a5). (Uxy: x is understood through y)

Chapter 4: Logic and Philosophy
§1: The Laws of Logic and their Bearers

As the title of this book shows, the central concern in this book is logical consequence, what follows from what. Relatedly, if not quite equivalently, we are asking about the theorems, or laws, of logic. Of what do the laws of logic hold?

Before Frege, it was common to think of logic as the laws of thought. The claim that logic summarizes rules of thought is ambiguous, and not very compelling on either sense. In the first sense, we take the claim to be descriptive: logic describes how we actually reason. On the descriptive interpretation, we could find the laws of logic by surveying opinions about correct inferences. Unfortunately, people's capacities for reasoning are limited. We make logical errors. On the descriptive interpretation of logic, if enough people make an error, it could become a rule of logic.

On the prescriptive interpretation of taking logic to be the rules of thought, logic tells us what thoughts should follow from which others. That claim is more plausible, but still not right. As Frege argued, any view that makes the laws of logic hold of thoughts makes logic subservient to psychology. Moreover, in many cases, we do not choose what to think about. The connections of our thoughts are influenced by environment and chemistry in ways that derivations in logic are impotent to redirect. If the laws of logic hold of ideas, then they are essentially subjective, whether we hold this claim to be descriptive or prescriptive.

In contrast, many more-recent philosophers, including Frege, believe that the laws of logic are objective. They are not about the connections of thoughts in our minds. Laws of logic are about entailments, whether formal or informal.

To say that the laws of logic are objective, though, does not determine what in particular they hold of. Consider a typical simple theorem of logic, the law of the excluded middle, EM.

EM $\quad\quad \alpha \lor {\sim}\alpha$

EM is a sentence in our logical metalanguage. It can be thought of as a schema, a rule for generating particular theorems of our object language: '$P \lor {\sim}P$'; '$Pa \lor {\sim}Pa$'; '$(\exists x)Px \lor {\sim}(\exists x)Px$'. Laws like EM are naturally expressed as schemas written in the metalanguage. They are not propositions themselves, but they tell you how to form certain propositions, or how to infer some propositions from others. We write EM as a way of saying that any substitution instance of that metalinguistic sentence is a tautology.

The substitution instances we form from logical laws are sometimes called logical truths. (Sometimes we call the incomplete sentence of the metalanguage a logical truth, as well.) 'It is raining, now' requires, for it to be true, some justification outside of logic. It must actually be raining in order for that sentence to be true. Logical truths require no justification outside of logic. They can be shown true using our semantics. And they can be derived regardless of any assumptions, or premises. In fact, they can be derived with no premises at all using the indirect or conditional methods of proof.

The logical truths are the theorems of the logical theory we are using. In addition to the law of the excluded middle, we have seen the law of non-contradiction LC (which is sometimes called, ironically, the law of contradiction).

LC $\quad\quad {\sim}(\alpha \bullet {\sim}\alpha)$

While EM and LC are among the most well-known logical truths, there are infinitely many theorems of all of our logical languages: **PL**, **M**, **F**, **FF**, and **S**. LT1 and LT2 are two more examples of schemas for producing logical truths of propositional logic:

LT1 $\alpha \supset (\beta \supset \alpha)$

LT2 $[\alpha \supset (\beta \supset \gamma)] \supset [(\alpha \supset \beta) \supset (\alpha \supset \gamma)]$

We began this section by asking about what the laws of logic hold. Now, it seems as if they hold of instances of schemas like EM, LC, and LT1 and LT2. But, what are these substitution instances? That is, what are we putting in the place of the 'α's?

One option for the instances is that the laws of logic hold of sentences. The word 'sentence' is ambiguous between sentence tokens and sentence types. To see the difference, consider how many sentences are in CD.

CD The cat dances. The cat dances.

There are two sentence tokens, but only one sentence type. The distinction between tokens and types holds of all sorts of objects. 'Mississippi'contains four letter types, but eleven letter tokens. The poster of Klimt's "The Kiss" hanging on a dorm-room wall is a token of the type which was originally instantiated by Klimt himself. A performance of Mahler's *Das Lied von der Erde* by your local orchestra is a token of the work.

The laws of logic do not hold merely of sentence tokens. Consider disjunctive syllogism.

DS $\alpha \vee \beta$
$\sim\alpha \quad / \beta$

In order for the conclusion of any instance of DS to follow from its premises, we have to substitute the same thing for α in both instances, and the same thing for β in both instances. But we can't substitute the same sentence token. If one token is in the first place, a different token would have to be put in the second place. So, the laws of logic can't be about sentence tokens.

Perhaps what we want are sentence types.

CI The cat either dances or sings.
She doesn't dance.
Therefore, she sings.

CI seems to be an instance of disjunctive syllogism. But there are different sentence types replacing α and different sentence types replacing β in DS. The first premise isn't even of the form '$\alpha \vee \beta$', on the surface. We can recast the first premise of CI so that it is more precisely in that form.

CI1 The cat dances or the cat sings.

Then 'the cat dances' replaces α in the major premise of DS and 'the cat sings' replaces β. Unfortunately, the conclusion, 'she sings', is still a different sentence type from 'the cat sings'. Similar remarks hold for what replaces α: 'she doesn't dance' has to be rewritten in order to look like the negation of 'the cat dances'.

While we can continue to recast CI so that we have exactly the same sentence types in precisely the right positions, we need not do so in order to conclude that it is a version of disjunctive syllogism. So, it looks like disjunctive syllogism doesn't hold of sentence types, either.

Fortunately there is another option. The third option is to take the substituends for our metalinguistic variables, i.e. that of which the laws of logic hold, to be propositions. To understand precisely what a proposition is, consider that CDE and CDS express the same proposition even though they are different sentences, both types and tokens.

CDE The cat dances.
CDS El gato baila.

Similarly, to adapt an example of Frege's, consider telling a friend that you are ill. You might express your condition using the sentence II.

II I am ill.

Suppose you ask your friend to call the doctor for you. She might use GI to explain her call to the doctor.

GI Gustav is ill.

II and GI are different sentences, but they express the same proposition.

Propositions are the meanings of sentence types. Notice that we do not see or hear propositions. We encounter sentence tokens. We infer the type from the presence of the token. By grasping the meaning of the type, we understand the proposition it expresses. Tokens are concrete, and types and propositions are abstract.

The abstractness of propositions does not make them psychological. Different people can think of the same proposition. My thoughts are in my mind, your thoughts are in your mind. Indeed, thoughts are tokens, like inscriptions. So propositions are not thoughts. They are the objects of thought. While our thoughts may share the same content, propositions are independent of any particular mind.

Propositions are naturally indicated by 'that', though they need not be expressed that way. The sentences 'the cat dances' and 'el gato baila' both express the proposition *that the cat dances*. We can see the use of that-clauses when we try to explain what a sentence means, as in MC.

MC 'snow is white' means-in-English that snow is white

Since propositions are meanings of sentences, we can see a proposition on the right side of MC. Notice that a that-clause is not a complete sentence: that snow is white; that 2+2=4; that the door is closed; that I am in Clinton NY. That-clauses are names of propositions.

Propositions can be used as subordinate clauses in a variety of other complex sentences. We can use a proposition in a question, a command, or in an expression of belief or desire.

Is it the case that snow is white? (Or 'Is snow white?')
Make it the case that the door is closed. (Or, 'Close the door'.)
I believe that 2+2=4.
I wish that I were in Puerto Rico.

In some cases, writing complex sentences using that-clauses makes them more awkward. But, it reveals their logical structure. Questions, commands, exclamations, beliefs, desires, and other complex forms contain references to that-clauses as their basic components.

In addition to being mind-independent, propositions are language-independent. They may be expressed by sentences of language, but they technically belong to no language. Some people think of propositions as states of affairs. What a sentence means is not in a language at all, just as the object to which a term refers is not a linguistic object.

Propositions are not the only kinds of abstract objects. Mathematical objects are also ordinarily taken to be abstract objects. Compare our knowledge of propositions with our knowledge of '2' and '2+2=4'. '2' is the name of a number, but is not the number itself, just as I am Russell, but not 'Russell'.

We see pizzas and frisbees, but we never see perfect geometric circles. Numbers and geometric objects are abstract: they are neither sensible objects nor ideas. They inhabit what Frege calls a third realm.

We learn about mathematics in part by using names of mathematical objects, and diagrams. Similarly, we learn of propositions from our interactions with sentence tokens. The sentence tokens we use in our proofs are names of propositions.

VR is a last example to help explain why logic deals with propositions, and not sentence types.

VR Visiting relatives can be annoying.

VR corresponds to a single sentence type, but it is ambiguous between two propositions: that it is annoying when relatives come to visit you and that it is annoying to visit one's relatives. If we try to substitute a sentence type in a rule of inference, like disjunctive syllogism, we are liable to generate false inferences because of the ambiguity. Technically, then, we substitute propositions for the schamatic variables in our laws of logic. Of course, we write tokens representing, or expressing, those propositions.

The view that the laws of logic hold of abstract objects called propositions might be called the traditional view, or the standard view. But, it has come under attack for the last century or so. Quine, who was much smarter about both logic and philosophy than I am, argues that the arguments for the existence of propositions are unsound, and that there are no such things as propositions. In fact, he calls intensions like propositions creatures of darkness. He argues that belief in meanings in general is a myth, the myth of the museum. Other philosophers, like the later Wittgenstein, are also skeptical about meanings. Some of the worry about meanings arises from a problem of access: how can we learn of objects in a distint third realm. Some philosophers, wary of propositions and the third realm, attempt to see the laws of logic holding of sentence types. But, types are still abstract objects and the problems of access to abstract objects remains.

Paper Topics

1. What are abstract objects? How do they differ from both concrete objects and ideas? Consider mathematical objects as well as propositions. You might also think about artworks including paintins and musical compositions. Consider an objection to the claim that there are abstract objects.

2. What are logical truths? How can we characterize a logical truth? What distinguishes a logical truth from other kinds of truths?

Suggested Readings

Bealer, George. "Propositions." In Jacquette.
Frege, "The Thought: A Logical Inquiry"
Katz, Jerrold. *The Metaphysics of Meaning*. The MIT Press, 1990.
Pap, Arthur. "The Laws of Logic." In Jacquette.
Quine, W.V. *Philosophy of Logic*, 2nd ed. Harvard University Press, 1986.
Read, Chapters 1 and 2.

§2: Disjunction, Unless, and the Sixteen Truth Tables

4.2.1. Logical Equivalence and Translation

In general, our logic is more fine-grained than natural language. We can use it to make careful distinctions, ones which are trickier to make in English. As Frege wrote, using logic is like looking at ordinary language through a microscope. But every logical language has its limits. One limit of propositional logic concerns its extensionality.

When comparing expressions of natural language, we can distinguish between intensional equivalence and extensional equivalence. Two phrases that are intensionally equivalent have the same meaning. Two phrases that are extensionally equivalent have the same reference. The difference between meaning and reference may be seen clearly in the phrases CH and CK.

CH	Creature with a heart
CK	Creature with kidneys

As a matter of biology, creatures have hearts if and only if they have kidneys. So, CH and CK pick out the same creatures; they have the same referents. But, CH and CK have different meanings. They are extensionally equivalent but intensionally different.

We provide the semantics for **PL** by giving truth conditions, using truth tables. As long as the truth conditions for two sentences are the same, we call the propositions logically equivalent. Our truth-functional logic does not distinguish between two logically equivalent propositions. Thus, our logic is extensional. Sentences with different intensions, like QF1 and QF2, may be translated identically.

QF1	Quine is an extensionalist and Frege is not.
QF2	It is not the case that either Quine is not an extensionalist or Frege is.

To see that QF1 and QF2 are extensionally equivalent even if they are intensionally distinct, let's regiment them and look at their truth tables.

Q	•	~	F
1	**0**	0	1
1	**1**	1	0
0	**0**	0	1
0	**0**	1	0

~	(~	Q	∨	F)
0	0	1	1	1
1	0	1	0	0
0	1	0	1	1
0	1	0	1	0

Whatever differences they might have in meaning, QF1 and QF2 are logically equivalent. Thus, as far as our truth-functional logic is concerned, we can use these two propositions interchangeably. They have the same entailments. They are consistent or inconsistent with the same propositions.

The notion of an intension, like the concept of a proposition, is controversial. For now, we will not pursue intensions. In contrast, the concept of logical equivalence is the central concept in the characterization of logic as extensional. The concept of logical equivalence allows us to clear up two related questions about translation. The first concerns the use of disjunction for 'unless'. The second concerns our use of inclusive disjunction for '∨'.

4.2.2. Unless and Exclusive Disjunction:

We ordinarily translate 'unless' using a $\vee$. In this section, using the concept of logical equivalence, I will explain why we do so.

Let's consider the sentence CR and think about what we want as the truth values of 'unless' in that sentence.

CR The car will not run unless there is gas in its tank.

We'll start by translating the 'unless' as a $\vee$, and constructing a standard truth table for the proposition.

~	R	$\vee$	G
0	1	**1**	1
0	1	**0**	0
1	0	**1**	1
1	0	**1**	0

Now, let's think about what we want as the truth values for the proposition expressed by CR.

The car runs	**The car will not run unless it has gas**	The car has gas
1		1
1		0
0		1
0		0

In the first row, the car runs and has gas, so the complex proposition CR should be true. In the second row, the car runs, but does not have gas. In this case, perhaps the car runs on an alternative fuel source, or magic. The proposition CR should thus be false in the second row.

In the third row, the car does not run, but has gas. Perhaps the car is missing its engine. This case does not falsify the complex proposition, which does not say what else the car needs to run. CR gives a necessary condition for a car to run (having gas), but not sufficient conditions. Thus CR should be considered true in the third row. In the fourth row, the car does not run and does not have gas. CR thus should be true in the fourth row.

So, from merely considering our desired truth values for the sentence, we get the following truth table for 'unless'.

The car runs	**The car will not run unless it has gas**	The car has gas
1	**1**	1
1	**0**	0
0	**1**	1
0	**1**	0

Notice that the truth table for 'unless' is precisely the same as the truth table for the $\vee$. Since the two truth tables are the same, we can use the $\vee$ to stand for 'unless'; it gives us precisely what we want.

Unfortunately, this felicitous result does not hold for all uses of 'unless'. Let's analyze LS the same way we analyzed CR.

LS Liesse will attend school full time unless she gets a job.

Liesse attends school	Liesse will attend school full time unless she gets a job.	Liesse gets a job
1		1
1		0
0		1
0		0

This time, let's work from the bottom up. In the last row, Liesse does not get a job but doesn't go to school. LS should be false, since it says that she will attend school unless she gets a job. In the third row, she gets a job, and doesn't go to school, and so the proposition should be true. In the second row, she attends school but doesn't get a job, and so the proposition should be true.

In the first row, Liesse gets a job but attends school anyway. What are your intuitions about the truth value of LS in this case?

In my experience, most people who have not studied formal logic take LS to be false in the first row. It's clear that if LS is true and Liesse does not get a job, then she will attend school. Most people also believe that if LS is true and Liesse does get a job, then she will not attend school. In this case, the truth table for LS should look as follows.

Liesse attends school	Liesse will attend school full time unless she gets a job.	Liesse gets a job
1	0	1
1	1	0
0	1	1
0	0	0

The truth table for ‘unless’ as used in LS seems to have the same truth conditions as exclusive disjunction, not for ∨.

α	∨	β
1	1	1
1	1	0
0	1	1
0	0	0

α	exclusive or	β
1	0	1
1	1	0
0	1	1
0	0	0

Unless thus appears to be as ambiguous as ‘or’, and in the same way: there’s an inclusive and exclusive ‘unless’. To regiment LS, we can use either ‘∼S ≡ J’ or ‘∼(S ≡ J)’, since they are logically equivalent to the truth table we constructed for the sentence..

∼	S	≡	J
0	1	**0**	1
0	1	**1**	0
1	0	**1**	1
1	0	**0**	0

∼	(S	≡	J)
0	1	1	1
1	1	0	0
1	0	0	1
0	0	1	0

We can thus think of the exclusive unless as a biconditional: Liesse will not attend school if, and only if, she gets a job. When faced with an unless, we ordinarily just take it to be a ∨. But, if we are concerned about getting the truth conditions precisely correct, then we have to decide whether the sentence functions more like CR, and so deserves the ∨, or more like LS, in which case we should write it with a ≡.

Here is another regimentation of LS, logically equivalent to the one we want: ‘(S ∨ J) • ~(S • J)’.

(S	∨	J)	•	~	(S	•	J)
1	1	1	**0**	0	1	1	1
1	1	0	**1**	1	1	0	0
0	1	1	**1**	1	0	0	1
0	0	0	**0**	1	0	0	0

We could, if we wished, introduce a new symbol for exclusive disjunction, say ‘XOR’ or ‘‘⊕’.

Inclusive Disjunction

α	∨	β
1	1	1
1	1	0
0	0	1
0	0	0

Exclusive Disjunction

α	⊕	β
1	**0**	1
1	**1**	0
0	**1**	1
0	**0**	0

But, we will not use ⊕, since we do not need it. If you have a sentence that you wish to regiment as an exclusive disjunction, you can use a proposition of the form ‘~α ≡ β’, or of any of the alternate forms.

Given two variables, there are sixteen possible distributions of truth values. We have labels for four: •, ∨, ⊃, and ≡. We can define the other twelve, using combinations of the five connectives. It is kind of a fun exercise and you might want to try it. As long as we can define all of the possibilities, though, it doesn’t matter which we take to be basic. We just have to be careful to translate our natural-language sentences to have the truth conditions that we want them to have. When translating unless, we ordinarily use the wedge for inclusive senses, and as the default translation. But we can use the biconditional (with one element negated) for exclusive senses.

§3: Conditionals

4.3.1. The Material Interpretation of the Natural-Language Conditional

There are lots of different kinds of conditionals in natural language.

A Indicative conditionals: If the Mets lost, then the Cubs won.
B Conditional questions: If I like logic, what class should I take next?
C Conditional commands: If you want to pass this class, do the homework.
D Conditional prescriptions: If you want a good life, you ought to act virtuously.
E Cookie Conditionals: If you want cookies, there are some in the jar.
F Subjunctive conditionals: If Rod were offered the bribe, he would take it.

The material conditional and its standard truth table are used for indicative conditionals. The material conditional is true unless the antecedent is true and the consequent is false.

α	$\supset$	β
1	1	1
1	0	0
0	1	1
0	1	0

The truth table is false only in the second line. Thus, we can think of the material conditional as saying that 'If α then β' is equivalent to 'Not (α and not-β)' because a material conditional is false only when the sentence replacing α is true and that replacing β is false.

The analysis of other kinds of conditionals seems to depend on our analysis of indicative conditionals. B, C, and D are not propositions as they stand, since they lack truth values. But we can parse them truth-functionally by turning them into indicatives.

B' If you like logic, then you take linear algebra next.
C' If you want to pass the class, you do the homework.
D' If you want a good life, you act virtuously

We can thus regiment B' - D' as material conditionals, just as we did for A. E is not really a conditional; it's a fraud.

The material conditional is probably the best truth-functional option for representing the conditional as it appears in English and other natural languages. But, the natural-language conditional is more complex than the material interpretation. Thinking about a proper treatment of conditionals quickly leads to important questions regarding the nature of scientific laws, and the ways in which they are confirmed or disconfirmed. Indeed, discussion of the proper treatment of conditionals is a central topic in the philosophy of science. Recent work in the logic of conditionals has also led to sophisticated modal extensions of classical logic, called conditional logics. Conditional logics are beyond the scope of this text, but are worth considering. Here, we will discuss a few of the subtleties of conditionals, and the challenges facing those who wish to pursue their proper logical treatment. In particular, conditionals of type F, subjunctive conditionals, pose interesting challenges.

4.3.2. Logical Truths and the Paradoxes of Material Implication

The material conditional creates what are called the paradoxes of material implication. To understand the paradoxes of material implication, one has first to understand the nature and importance of logical truth. A logical truth is a special sentence of a logical system, one which is true on all interpretations. The logical truths are thus the theorems of a system of logic. In an axiomatic system, like Euclidean geometry or a formal treatment of Newtonian mechanics, we choose a small set of privileged sentences that we call axioms. The axioms define the system. In many cases, we insist that the axioms be obvious and uncontroversial. The theorems of a formal system are the statements that are provable from the axioms. Some sentences of propositional logic are theorems. These statements are the logical truths.

We identify a system of logic, indeed any formal system, with its theorems. Competing theories have different theorems. Two theories with different axioms, or assumptions, can turn out to be equivalent, if they yield the same theorems.

> Demarcate the totality of logical truths, in whatever terms, and you have in those terms specified the logic (Quine, *Philosophy of Logic*, p 80.)

In order to know which system of logic we are using, which assumptions that system makes, we look at the logical truths of the system. To get a feel for logical truths, consider two.

F $\quad P \supset P$
G $\quad [(P \supset (Q \supset R)] \supset [(P \supset Q) \supset (P \supset R)]$

F and G have a natural obviousness that properly characterizes a theorem of logic, which is supposed to be the most obvious of disciplines. Many other tautologies are also obvious.

Among the paradoxes of material implication are statements of forms H, I and J. We call them paradoxes, a name which is probably too strong, beacuse they turn out to be logical truths even though they are not obvious.

H $\quad \alpha \supset (\beta \supset \alpha)$
I $\quad \sim\alpha \supset (\alpha \supset \beta)$
J $\quad (\alpha \supset \beta) \vee (\beta \supset \alpha)$

The paradoxes of material implication are both unobvious and have awkward consequences. H says, approximately, that if a statement is true, then anything implies it. For, the truth table for the material conditional is true on every line in which the consequent is true. So, K is true, on the material interpretation.

K $\quad$ If Martians have infra-red vision, then Obama is president of the United States in 2011.

Schema I says that if a statement is false, its opposite entails any other statement. So L is true on the material interpretation: since the antecedent of L is false, even an absurd consequent follows from it.

L $\quad$ If Bush is president of the United States in 2011, then Venusians have a colony on the dark side of Mercury.

Lastly, J says that for any statement, β, either any other statement entails it, or it entails any statement. Every statement must be either true or false. If a given statement is true, then, as in H, any statement entails it. If a given statement is false, then, as in I, it entails any statement. So M is true, according to the material interpretation of the conditional.

M Either 'Neptunians love to wassail' entails 'Saturnians love to foxtrot' or 'Saturnians love to foxtrot' entails 'Neptunians love to wassail'.

Indeed, M is not only true, but a law of logic. Further, either N or O is true.

N 'It is raining' entails 'Chickens are robot spies from Pluto'.
O 'Chickens are robot spies from Pluto' entails 'It is raining'.

If it is raining, then O is true (whatever the case is about chickens). If it is not raining, then N is true. According to a further law of logic, called excluded middle, either it is raining or it is not raining. So, one or the other of N or O must be true even though they are both absurd sentences.

In sum, the paradoxes of the material conditional are two kinds of awkward results. First, statements of the form of H, I, and J (M, for example) are laws of logic that are not obviously true. Second, statements like K, L, and either N or O are true, given the truth values of their component propositions, even we do not intuitively see them as true.

4.3.3. Dependent and Independent Conditionals

The paradoxes of material implication show that there is something funny going on with the ⊃. To begin diagnose the problems with the material conditional, let's distinguish between dependent and independent conditional statements.

A *dependent conditional* has a connection between its antecedent and consequent.

I'll leave exactly what I mean by a connection unstated, here, but here are some examples. P-S are dependent conditionals.

P If it is raining, then I will get wet.
Q If I run a red light, then I break the law.
R If the car is running, then it has fuel in the tank.
S If I were to jump out of the window right now, I would fall to the ground.

The material interpretation seems acceptable for dependent conditionals like P-S. Even when the antecedents are false, connections between the antecedents and consequents hold. Recall the example from §1.5.

T If you paint my house, I will give you five thousand dollars.

If the antecedent of T is not true, if you do not paint my house, we can take the conditional to be true as a standing offer. P is true because whether or not it is actually raining, I will get wet if it is. Q is true because whether or not I run a red light, the connection between doing so and breaking the law remains.

In contrast, consider some independent conditionals.

An *independent conditional* lacks the connection we find in a dependent conditional

K, L, N and O, awkwad instances of the paradoxes of the material conditional above, are independent conditionals. So are U-X.

U If 2+2=4, then cats are animals.
V If 2+2=4, then cats are robots.
W If pigs fly, then Utica is near Rome.
X If pigs fly, then Utica is the capital of Canada.

The material interpretation is awkward for independent conditionals. Since '2+2=4' is true and 'cats are animals' is true, U is true. Since '2+2=4' is true and 'cats are robots' is false, V is false. Since 'pigs fly' is false, W and X are both true. All of these results seem counter-intuitive for the natural-language conditional. I am hesitant to pronounce at all on their truth values. The material interpretation of the conditional thus seems acceptable for dependent conditionals. But, it is only uncomfortably applied to independent conditionals.

The paradoxes of material implication are awkward because they hold for any values of the propositional variables, whether the relation is dependent or independent. Still, we might accept the material analysis of the conditional merely for the benefits it yields to the dependent conditional. The material interpretation of the conditional returns a truth value for any conditional combination of propositions. It allows us to maintain the truth functionality of our logic: the truth value of any complex sentence is completely dependent on the truth value of its component parts. The dependent conditional is much more common than the independent one, anyway, and most people don't have strong feelings about the truth values of sentences like U-X. The paradoxes of material implication may thus just be seen as the price we have to pay to maintain truth-functionality.

One response to the paradoxes of material implication which you might be considering is to find a different truth table for the conditional. It's worth a moment to see that this response is not productive. We will do so in two stages, first looking at the first two rows of the truth table for the material conditional, and then at the second two rows.

4.3.4. Nicod's Criterion and the First Two Rows of the Truth Table

The first two lines of the truth table for the material conditional, which are not counterfactual, look fine, especially in dependent conditionals like P-T. They represent what is known as Nicod's criterion for confirmation of a scientific claim.

Many scientific laws are conditional in form. Nicod's criterion says that evidence will confirm a law if it satisfies both the antecedent and consequent of such a law. It also says that evidence will disconfirm a law if it satisfies the antecedent, but fails to satisfy the consequent. Let's take a sample law, Coulomb's law, which says that the force on two particles is proportional to the absolute value of product of the charges on each particle (q_1 and q_2) divided by the square of the distance between them (r).

CL $F = k\ |q_1q_2|/\ r^2$.

We analyze CL as a claim that *if* two particles have a certain amount of charge and a certain distance between them, *then* they have a certain, calculable force between them. We take evidence to confirm the law if it satisfies the antecedent and the consequent of that conditional. We take evidence to disconfirm the law if it were to satisfy the antecedent and falsify the consequent. If we were to find two particles which did not have the force between them that the formula on the right side of Coulomb's Law says should hold, and we could not find over-riding laws to explain this discrepancy, we would seek a revision of Coulomb's Law.

To take a simpler example, consider the claim that all swans are white. We may analyze that claim as, 'if something is a swan, then it is white'. When we find a white swan, which satisfies the antecedent and the consequent, it confirms the claim. If we were to find a black swan, which satisfies the antecedent but falsifies the consequent, then it would disconfirm the claim.

According to Nicod's criterion, instances which do not satisfy the antecedent are irrelevant to confirmation or disconfirmation. A white dog and a black dog and a blue pen have no effect on our confidence in the claim that all swans are white. Call a conditional in which the antecedent is false a counterfactual conditional. Nicod's criterion thus says nothing about counterfactual conditionals.

We are considering alternatives to the material interpretation of the conditional. The point of mentioning Nicod's criterion was to say that we should leave the first two lines of the truth table alone.

4.3.5. The Immutability of the Last Two Rows of the Truth Table for the Material Conditional

Given the first two rows, there are three possibilities for the third and fourth lines of the truth table for the conditional that are different from the material interpretation.

Option A

α	⊃	β
1	1	1
1	0	0
0	1	1
0	0	0

Option B

α	⊃	β
1	1	1
1	0	0
0	0	1
0	1	0

Option C

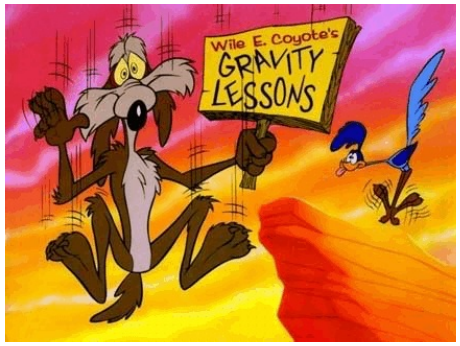
α	⊃	β
1	1	1
1	0	0
0	0	1
0	0	0

Option A gives the conditional the same truth-values as the consequent. It thus makes the antecedent irrelevant. Option B gives the conditional the same truth-values as a biconditional. Option C gives the conditional the same truth-values as the conjunction. The conditional seems to have a different role in natural language from either the biconditional or the conjunction. Thus, the truth table for the material conditional is the only one possible with those first two lines that doesn't merely replicate a truth table we already have.

To see the problem more intuitively, consider again a good counterfactual dependent conditional like S.

S If I were to jump out of the window right now, I would fall to the ground.

Option A says that S is falsified when I don't jump out the window and I don't fall to the ground. Options B and C say that S is falsified when I don't jump out of the window and I do fall to the ground. But, neither case seems to falsify S, as it is intended. The only time that S is falsified, as on Nicod's criterion, is in the second line of the truth table, when I jump out of the window and, like the Coyote for just a moment after he races off a cliff, I hang in the air.



It looks like we have to stick with the original truth table, if we want the conditional to be truth-functional.

4.3.6. Subjunctive and Counterfactual Conditionals

We have been considering whether and to what extent the material conditional stands for the natural-language conditional. Our worries about independent indicative conditionals were allayed by two considerations. First, we don't have strong feelings about the truth values of many of those sentences, like U-X, which seem rare and deviant. Second, our desire to maintain truth-functionality entails that we have to give all complex propositions truth values on the basis of the truth values of their component sentences, and the material conditional is the only option that respects our strong feelings about the first two rows of the table. The first two rows of the truth table for the material conditional capture our intuitions about dependent conditionals, as well. Despite worries about the paradoxes of the material implication and the oddities of material interpretations of independent conditionals, the material interpretation seems to be the best way to maintain truth-functionality.

Unfortunately, more problems beset the material conditional. Subjunctive conditionals, especially in their counterfactual interpretations, raise further problems. Consider again E.

E If Rod were offered the bribe, he would take it.

If its antecedent is true, we know how to evaluate E. If Rod takes the bribe, then E is true; if he refuses the bribe, then E is false. According to the material interpretation of the conditional, if Rod is never offered the bribe, then E is true. In the case of E, this result might be acceptable. But, there are cases in which a conditional with a false antecedent should be taken as false. Compare S, which is a subjunctive conditional similar to E, with S'.

S If I were to jump out of the window right now, I would fall to the ground.
S' If I were to jump out of the window right now, I would flutter to the moon.

According to the material interpretation, S and S' are true, since I am not jumping out of the window. But, S is true and S' is false.

The difference between S and S', and its inconsistency with the material interpretation of the conditional, has come to be known as the problem of counterfactual conditionals. Nelson Goodman, in his famous paper on the problem, contrasts Y with Y'.

Y If that piece of butter had been heated to 150°F, it would have melted.
Y' If that piece of butter had been heated to 150°F, it would not have melted.

Let's imagine that we never heat that piece of butter, so that Y and Y' both contain false antecedents. According to the material interpretation, since the antecedents of Y and Y' are both false, or counterfactual, both sentences come out as true. But, it seems that they should be taken as contraries. They can't both be true. Indeed, just as we want to call S true and S' false, we want to call Y true and Y' false.

We have already seen that there are no good options for alternative truth tables. If we want to distinguish among counterfactual conditionals, we may have to quit thinking of the natural-language conditional as a truth-functional operator.

4.3.7. Non-Truth-Functional Operators

We might resolve the tension between Y and Y' by claiming that 'if...then...' has two meanings. The, let's say, logical aspects of the natural-language conditional may be expressed by the truth-functional '$\supset$' , encapsulated by the truth table for the material conditional. Other aspects of the natural-language conditional might not be truth-functional at all. We could introduce a new operator, strict implication, $\Rightarrow$, to regiment conditionals whose meaning is not captured by the material interpretation. Statements of the form '$\alpha \supset \beta$' could continue to be truth-functional, even though statements of the form '$\alpha \Rightarrow \beta$' would be non-truth-functional. So, consider again, sentence U.

U If 2+2=4, then cats are animals.

We could regiment it as '$T \supset C$' in the standard way. We could, alternatively, regiment it as '$T \Rightarrow C$'. '$T \Rightarrow C$' would lack a standard truth-value in the third and fourth rows. We could leave the third and fourth of the truth table rows blank, neither true nor false. Or, we could add a third truth value, often called undetermined, or indeterminate. Such a solution would leave many conditionals, especially counterfactual conditionals, without truth values. Introducing $\Rightarrow$ would entail giving up our neat bivalent semantics for propositional logic.

Another option, deriving from the early-twentieth-century logician C.I. Lewis and gaining popularity in recent years, is to interpret strict implication modally. Lewis defined '$\alpha \Rightarrow \beta$' as '$\Box(\alpha \supset \beta)$'. The '$\Box$' is a modal operator. There are many interpretations of modal operators. They can be used to construct formal theories of knowledge, moral properties, tenses, or knowledge. For Lewis's suggestion, called strict implication, we use an alethic interpretation of the modal operator, taking the '$\Box$' as 'necessarily'. So, on the modal interpretation of conditionals, a statement of the form '$\alpha \Rightarrow \beta$' will be true if it is necessarily the case that the consequent is true whenever the antecedent is.

Modal logics are controversial. Some philosophers believe that matters of necessity and contingency are not properly logical topics. Other philosophers worry that our ability to know which events or properties are necessary and which are contingent is severely limited.

One advantage of introducing a modal operator to express implication is that it connects conditional statements and scientific laws. A scientific law is naturally taken as describing a necessary, causal relation. When we say that event A causes event B, we imply that A necessitates B, that B could not fail to occur, given A. To say that lighting the stove causes the water to boil is to say that, given the stability of background conditions, the water has no choice but to boil. Thus, we might distinguish the two senses of the conditional by saying that material implication represents logical connections, where strict implication attempts to regiment causal connections.

4.3.8. Counterfactual Conditionals and Causal Laws

As we have seen, the natural-language conditional often indicates a connection between its antecedent and its consequent. Consider again the dependent conditionals P-T.

P If it is raining, then I will get wet.
Q If I run a red light, then I break the law.
R If the car is running, then it has fuel in the tank.
S If I were to jump out of the window right now, I would fall to the ground.
T If you paint my house, I will give you five thousand dollars.

The connection in Q is mainly conventional. T refers to an offer or promise. But, P, R, and S are fundamentally causal, depending on most basic scientific laws. Our investigation of the logic of the conditional has taken us into questions about causation.

α If this salt had been placed in water, it would have dissolved.

α indicates a dispositional property of salt, one that we use to characterize the substance. Other dispositional properties, like irritability, flammability, and flexibility, refer to properties interesting to scientists. Psychological properties, like believing that it is cold outside, are often explained as dispositions to behave, like the disposition to put on a coat or say, "It's cold." Contrast α with β,

β This marble counter is soluble in water.

If we never place the counter in water, then β comes out true on the material interpretation. To be flammable is just, by definition, to have certain counterfactual properties. These pajamas are flammable just in case they would burn if subjected to certain conditions. The laws of science depend essentially on precisely the counterfactual conditionals that the logic of the material conditional gets wrong.

Goodman argues that the problem with counterfactual conditionals is that they are not merely logical relations. The problem of giving an analysis of the logic of conditionals is intimately related to the problem of distinguishing laws from accidental generalizations, as the comparison of γ with δ shows.

γ There are no balls of uranium one mile in diameter.
δ There are no balls of gold one mile in diameter.

The explanation of γ refers to scientific laws about critical mass. If you gather too much uranium in one spot, it explodes. The explanation of δ, in contrast, is merely accidental. It is entirely possible that we could gather that much gold together, while it is impossible to gather the same amount of uranium. In order to know that difference, though, you must know the laws which govern the universe.

Goodman's claim is that the problem of distinguishing γ from δ, the problem of knowing the laws of nature, is inextricably linked to the problem of understanding the logic of the natural-language conditional. We may use conditionals as truth-functional connectives, sometimes. More commonly, especially in counterfactual cases, we use them to state connections between antecedents and consequents. So, a conditional will be true if the relevant connections hold among the antecedent and the consequent. It is false if such connections do not.

> A counterfactual is true if a certain connection obtains between the antecedent and the consequent. But...the consequent seldom follows from the antecedent by logic alone (Goodman, "The Problem of Counterfactual Conditionals" 7-8).

Consider α, or Goodman’s sentence ε.

ε If that match had been scratched, it would have lighted.

For either α or ε to be true, there have to be a host of other conditions satisfied.

> We mean that conditions are such - i.e. the match is well made, is dry enough, oxygen enough is present, etc. - that “That match lights” can be inferred from “That match is scratched.” Thus the connection we affirm may be regarded as joining the consequent with the conjunction of the antecedent *and other statements that truly describe relevant conditions* (Goodman, “The Problem of Counterfactual Conditionals” 8, emphasis added).

When we assert a conditional, we commit ourselves to the relevant related claims. But, to understand the claims to which we are committed, we must understand the relevant connections between the antecedent and the consequent. We must understand the general laws connecting them.

> The principle that permits inference of ‘That match lights’ from ‘That match is scratched. That match is dry enough. Enough oxygen is present. Etc.’ is not a law of logic but what we call a natural or physical or causal law (Goodman, “The Problem of Counterfactual Conditionals” 8-9).

In order to infer the consequent of α, for example, from its antecedent, we need to presume causal laws governing the dissolution of salt in water. In order to infer the consequent of ε from its antecedent, we need to presume causal laws about the lighting of matches. Goodman thus argues that a proper analysis of counterfactual conditionals would include two elements.

1. A definition of the conditions that are relevant to the inference of a consequent from an antecedent.
2. A characterization of causal laws.

We have gone far from just understanding the logic of our language. We are now engaged in a pursuit of the most fundamental features of scientific discourse. Distinguishing between S and Y, or between Z and Z', in contrast to the material interpretation, would import extra-logical features into our logic. While we believe that Y and Z' are false, our reasons for that belief do not seem, now, to be a matter of logic. The reasons we think that they are false are due to the laws of physics. If we were living on a planet with very little gravitational force, but on which buildings had limited force fields that kept us tethered to the ground inside, it might indeed be the case that if I jumped out of the window, I would fly to the moon, rather than fall to the ground.

We really want our logic to be independent of all the extra-logical facts. We don’t want to import the physical facts into our logic, since we want our logic to be completely independent of the facts about the world. Thus, we rest with the material interpretation of the natural-language conditional, giving up hope for a truth-functional analysis of the causal conditional, and remembering that the natural-language conditional represents a strictly logical relation.

Paper Topics

1. Contrast the following pair of counterfactual conditionals.

> If bin Laden didn't plan the 9-11 attacks, then someone else did.
> If bin Laden hadn't planned the 9-11 attacks, then someone else would have.

The antecedents and consequents of these statements are nearly identical, but, our estimations of the truth values and semantics of U and U' are different. Discuss the similarities and differences among these sentences. Can we use the material conditional for this example? Are there other options? See Bennett and Jackson for discussions of a relevantly similar pair of sentences.

2. Consider the following inference.

> If this is gold, then it is not water-soluble.
> So, it is not the case that if this is gold then it is water-soluble.

Intuitively, this argument seems valid. But, if we regiment the argument in a standard way, we get an invalid argument. Discuss this problem in the light of the discussion of the material conditional. For possible solutions, you might look at Lewis and Langford 1932; Priest 2008; or Goodman's work.

3. In relevance logic, we insist that for a conditional to be true, its antecedent and consequent must be appropriately related. People working on relevance logics are mostly following C.I. Lewis's suggestion concerning strict implication. See Priest 2008.

4. Lewis on strict implication

5. The philosopher Paul Grice, responding in part to the problems of the conditional, distinguished between the logical content of language, and other, pragmatic, features of language. In addition to Grice's paper, Fisher, Priest, and Bennett all have useful discussions of Grice's suggestion.

6. Connections to three-valued logics

7. Lewis Carroll's paper, "A Logical Paradox"

8. Goodman, and the relation between conditionals and scientific laws. Hempel.

9. Frank Jackson and David Lewis have extended treatments of conditionals

Suggested Readings

Bennett, Jonathan. *A Philosophical Guide to Conditionals*. Oxford University Press, 2003.

Carroll, Lewis. "A Logical Paradox." *Mind* 3:436-8, 1894.

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Harper, William, Robert Stalnaker, and Glenn Pearce. *Ifs*. Reidel, 1981. Lots of advanced papers, some quite technical.

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Lewis, David. *Counterfactuals*. Blackwell, 1973.

Lewis, C.I., and Langford, C.H. *Symbolic Logic*. New York: Century Company, 1932. Reprinted by Dover Publications (New York), 1959.

Priest, Chapters 1, 4, and 5.

Read, Chapter 3.

Weiner, Joan. "Counterfactual Conundrum." *Nous* 13.4: 499-509, 1979.

§4: Syntax, Semantics, and the Chinese Room

4.4.1. Theories of the Mind

We have mainly been thinking of logic as a language for formally representing our beliefs and inferences. We can also use it to program computers and to simulate human reasoning with machines. The electronic circuitry in computers follows logical laws and logic is used to plan and program computers. In science fiction stories, we are often faced with robots or androids who seem to be just like human beings. Neuroscientists modeling brain activity and function use computers to model human cognitive processing. These uses of logic lead us naturally to ask the question, "What is a mind?" Can machines like robots have minds? Similarly, we could ask the question whether animals have minds.

There are four prominent, general theories about the nature of mind: dualism, behaviorism, identity theory, and functionalism. I will discuss the first three theories briefly, and the major difficulties they face. Then, we will focus on the fourth. The distinction we have made formally between the syntax and the semantics of a theory supports a surprising claim about the nature of minds and about the possibility of artificial intelligence.

Dualism in the philosophy of mind is the theory that there are two kinds of things, or substances, in the world: there are bodies and there are minds. The dualist believes that minds are non-physical substances. We sometimes call dualist minds souls; they are the seat of thought. Someone who believes that the soul can live past the death of the body is a dualist. Among prominent dualists in the history of philosophy are Plato and Descartes.

According to dualists, our minds are somehow attached to our bodies, while our bodies are alive, while being independent of them. Most dualists believe that the mind can live past the death of the body. Some dualists believe that the mind exists prior to the body. The central problem for dualism is the problem of interaction: how does an immaterial substance interact with a physical substance?

It is easy to see how two physical objects can affect one another by impact. When a swung bat hits a thrown ball, it transfers some of its momentum to the ball. Bodies can also affect each other at a distance. Magnets create fields of attractive force. The Earth's gravitational force keeps the moon in orbit; the moon's gravitational force creates the ocean tides on Earth. All of these cases are of physical objects affecting other physical objects. If the mind is an immaterial soul, then it seems impossible for it to have any physical effects in the world. How can my thought that I would like a milk shake lead me to drink a milk shake?

The dualist must explain some way for the mind to communicate with the body. The communication can not be strictly physical since the mind is isolated from the physical world. Similarly, the communication can not be strictly mental since the physical world is isolated from the mental world. If there are two distinct kinds of substances, it seems impossible that they would be able to communicate.

In opposition to dualists, before the twentieth century, there were two different kinds of monists. Idealist monists, like Leibniz and Berkeley, claimed that the physical, material world is illusory and that the only real things are mental: ideas and their (non-physical) thinkers. I won't spend time on idealist views. Materialist monists, like Hobbes, denied the existence of minds and claimed that only material substances exist: everything is bodies and there are no immaterial souls. The central problem for the materialist monist, historically, is that it seemed unlikely that something as complex and private and ineffable as human consciousness could be the product of physical interactions.

> [P]erception and that which depends upon it are inexplicable on mechanical grounds, that is to say, by means of figures and motions. And supposing there were a machine, so constructed as to think, feel, and have perception, it might be conceived as increased in size, while keeping the same proportions, so that one might go into it as into a mill. That being so, we should, on examining its interior, find only parts which work one upon another, and never anything by which to explain a perception (Leibniz, *Monadology* §17).

Until the twentieth century, the possibility of explaining human consciousness seemed to many people to be as absurd as it did to Leibniz. How could there be physical explanations of pains and pleasures and perceptions and thoughts? But progress in science began to make the possibility of explaining minds in physical terms increasingly plausible. Behaviorism was the first serious materialist theory developed in the twentieth century. The behaviorist says that minds are just behaviors. To say that someone is hungry (a mental state) is just to say that s/he is disposed to go to have a meal or a snack (both behaviors). My desire for a milk shake is not some internal thought about a milk shake; it is just my behavior, or my predisposition for behavior, around milk shakes. To say that someone is in pain is just to say that s/he is likely to cry or scream or in some other way express pain behavior.

Behaviorism was favored by many psychologists. By defining mental states as dispositions to behave, mental states became accessible to scientists, legitimate objects of serious empirical study unlike immaterial souls. The behaviorist thus avoids the problem of interaction. Behaviors are physical. If mental states are just behaviors or dispositions to behave, then they can be explained in terms of physical facts.

Unfortunately, the behaviorist provides an anemic account of our internal states. For example, consider two chess players. The first player stares inscrutably at a chess board for fifteen minutes, thinking quietly and unnoticeably about various different possible moves. After fifteen minutes, the first player makes a move. Another chess player, faced with the same board, also stares at it for fifteen minutes. But, the second player is thinking about restaurants for dinner after the match. After fifteen minutes, the second player makes the same move as the first player, but without thinking about any other possible moves. Both players exhibited the same behavior. But, they did so as a result of very different mental processes. The behaviorist has little ability to differentiate the mental states of the two players, since they both exhibited the same behaviors.

The third prominent theory of mind, identity theory, was an attempt to recapture internal mental states without succumbing to dualism. The identity theorist identifies minds with brains: mental states are just human brain states. Identity theory thus differentiates our two chess players on the basis of their quite different neural processes. Since the two players had different neural firings, they had different thoughts even though their behaviors were identical. Identity theory is supported, obviously, by neurological research. The more we understand about the brain, the less compelling Leibniz's claim that it could not support thought appears to be. Progress in brain science has refined our understanding of different mental capacities and states remarkably over the last century.

But our increased understanding of brain states and their correlations with certain mental states has not supported the identity theory. One serious problem with identity theory is its chauvinism. According to identity theory, only human beings can have minds, since only human beings have human brains. Imagine that we meet an alien from another planet made out of a different substance, say silicon. The aliens brain, let's suppose, has a radically different organization from a human brain. Further, let's suppose that the alien behaves and interacts with us as if it were human. We would surely grant that the alien has a mind. But, if mental states are brain states, as the identity theory says, then the alien, lacking a human brain, could not have a mind. More importantly, to return to our original question, androids, machines that act like people, could not be conscious by definition: they lack human brains.

In response to the difficulties with these three theories of the mind, many philosophers defend a fourth theory: functionalism. Most functionalists agree with identity theorists and other materialists that there are no immaterial souls. Most functionalists also agree that there is more to our mental life than our behavior. But the functionalist identifies minds with the brain's processing, rather than the brain itself. According to functionalism, anything that behaves like something with a mind and that has internal processes that map onto our internal processes, has a mind. Since computers do not have brains, the identity theorist says that they can not think. Functionalists argue that what is important about minds is the software, rather than the hardware: the mind is the software of the brain. Functionalists are thus sympathetic to the claim that machines can think, that there can be artificial intelligence (AI). According to the functionalist defender of AI, mental states can be instantiated in different kinds of physical bodies. They could even be instantiated in an immaterial substance. What's important is the processing of information, not any physical basis of that process. Functionalism thus avoids the chauvinism of identity theory. Any kind of thing can be a mind: a human being, a computer, an alien, an immaterial soul, as long as it processes information, responds to inputs, and produces behaviors in ways that are indicative of the kinds of acts that we call mental.

> Saying Deep Blue doesn't really think about chess is like saying an airplane doesn't really fly because it doesn't flap its wings (Drew McDermott, ftp://ftp.cs.yale.edu/pub/mcdermott/papers/deepblue.txt).

Functionalism might be seen as an abstract form of identity theory. Compare yourself to a sophisticated android, one that is built with the same structure as yourself, but out of different materials. You can see rainbows and taste strawberries. The android can see rainbows and taste strawberries. You can have a toothache, and the android can have a toothache. The details of what happens in your brain when you have a toothache will be different from the details of what happens in the android's brain. Your pain is in a human brain. The android's pain is in its silicon circuitry. But, the functional organization will be the same: the same kinds of inputs will produce the same kinds of outputs, with the same sorts of intermediate processes. For the functionalist, you and the android each have a pain. What makes the mental state such that you can each have it is its functional organization, not its material instantiation.

Functionalism is both consistent with a materialist view of the world and subtle enough to accommodate both internal processes (like the identity theorist) and behavioral correlates of psychological states (like the behaviorist). Thus, it has been popular with computer scientists eager to find profound implications of their work. Sometimes, these scientists can make extravagant claims. For example, the computer scientist John McCarthy claims that even the simplest machines can have beliefs. He argues that a thermostat has three beliefs: that it is too cold, that it is too hot, and that it is just right. We might want to take McCarthy's use of 'belief' as metaphorical, not literal. When my wife says that the mosquitos believe that she is tasty, we are best interpreting her words as metaphors. Most of us don't think that mosquitos have conscious beliefs, but we explain their behavior simply by ascribing beliefs to them. On the other hand, if someone says that a chimp or a dolphin has beliefs, we don't know whether to take such a statement literally or metaphorically, or to what degree. But McCarthy does not intend his claim about the beliefs of machines to be understood metaphorically. Defenders of artificial intelligence believe that there is no important difference between human beliefs and machine beliefs.

For the functionalist, whether you are in a particular mental state depends on both external, verifiable factors and internal factors like whether or not you are actually perceiving the rainbow or feeling the toothache. Those latter criteria are, like the hypothesis that other human beings are sentient, not amenable to external verification. I have no way of knowing for sure whether or not you are (or any other human being is) a carefully-crafted robot. You can say that you are not a robot, but you might just be constructed to say that, like a child's talking doll. I know precisely what the conditions for you to be sentient are. For you to be conscious is for you to be relevantly like me. But, I can not verify, or experience for myself, the contents of your mind.

One of the main problems with functionalism and AI involves the qualitative aspect of conscious experience. I know what mangoes taste like, independently of their chemical properties. I assume that you do too, despite my not being able to verify that you do. It seems unlikely that an android would experience the sweet taste of a mango in the same way that you and I experience it. We will not explore that problem, here, though. Our interest in functionalism and AI depends on a famous objection to the possibility of AI from a philosopher named John Searle. Searle's worry concerns a different mental property, which we can call intentionality.

4.4.2. Searle and Strong AI

Searle presents an argument against functionalism and artificial intelligence based on the distinction between syntax and semantics. His argument is directed against a strong AI thesis. A weak AI thesis is just the unobjectionable claim that machines built to perform tasks that humans perform can give us some insight into the nature of our thought. Weak AI is uncontroversial, except for the most enthusiastic dualists. Proponents of AI are committed to a stronger thesis.

Cheap calculators can now perform very complicated tasks much more quickly than even the smartest humans. Machines are already able to do many tasks that once were inconceivable, including proving mathematical theorems that require more computation than humans can perform. Better machines may approach or overtake human skill in other areas as well. The strong AI claim is that computers with such skills actually have minds. It is the same as McDermott's claim about Deep Blue. The claim entails that we need not know about the structure of the brain in order to know about the structure of the mind. All we need in order to have a mind is to simulate the behavior, along with some plausible internal causes of that behavior. To understand minds, according to Strong AI, we only need to understand computer models and their software. Searle's characterization of strong AI is the same as our characterization of functionalism in terms of computers; the mind is the software of the brain.

The first thing to notice about computers and their software is that they work according to purely formal, syntactic manipulation. The syntax of a program or system of formal logic concerns its form, or shape. Our rules for wffs are syntactic. The semantics of a system or program concerns the meanings of its terms. When we interpret a set of propositional variables as meaning something, we are providing a semantics, as when we give a translation key for a formal argument. It will be useful, both now and later, to make a clear distinction between syntax and semantics.

4.4.3. Syntax and Semantics

This text is centrally focused on constructing and using formal systems of logic. Whenever we introduce a formal system of logic, we introduce two languages: an object language and a metalanguage. The object language is the language that we are studying. The metalanguage is the language we use to study the object language. The rules for well-formed formulas are written in the metalanguage, but they are about how to construct the object language. The rules for forming wffs are syntactic. Similarly, the rules for constructing truth tables, indeed the truth tables themselves, are written in a metalanguage. That's why we use 1 and 0, which are not symbols of our object language. The rules for assigning truth values are semantic rules.

Whenever one constructs a formal language, one provides both a syntax and a semantics for that language. The syntax tells how the formulas are constructed. The semantics tells how to interpret the formulas. Inference rules and rules of equivalence are also specified syntactically. They hold for any interpretation of the formulas, which makes them both powerful and uncontroversial.

Separating the syntax of our language from its semantics allow us to treat our formal languages as completely uninterpreted, or topic-neutral. We can play with the symbols, according to the rules we specify, as if they were meaningless toys. We can interpret our languages variously, comparing interpretations in order to see the properties of the language itself clearly. Frege, indeed, was motivated specifically by the possibility of specifying a syntactic criterion for logical consequence. He wanted to ensure that some odd results which had arisen in mathematics in the nineteenth century were not illegitimate. He wanted to ensure that all deductions are secure, and that we do not implicitly smuggle into our results unjustifiable interpretations. We, like Frege, want to make sure that we do not presuppose a hidden, implicit premise.

The preface to Frege's *Begriffsschrift*, the title of which means concept-writing, makes his motivation clear.

> So that nothing intuitive could intrude [into our concept of logical consequence] unnoticed, everything had to depend on the chain of inference being free of gaps. In striving to fulfil this requirement in the strictest way, I found an obstacle in the inadequacy of language: however cumbersome the expressions that arose, the more complicated the relations became, the less the precision was attained that my purpose demanded...The present *Begriffsschrift*...is intended to serve primarily to test in the most reliable way the validity of a chain of inference and to reveal every presupposition that tends to slip in unnoticed, so that its origin can be investigated.

By separating the syntax of logic, its formation and derivation rules, from its semantics, its interpretations and our ascriptions of truth and falsity, we are attempting to fulfil Frege's dream of a secure theory of logical consequence.

4.4.4. The Chinese Room

Computers, in their most basic form, contain a complete list of possible states of the system, and possible inputs, and the output, all specifiable syntactically. The actions of a computer are completely determined by its algorithm, or set of rules. An algorithm is just a list of instructions, a procedure. Computer programs are algorithms; cooking recipes are algorithms. Recipes generally just give simple, linear instructions. An algorithm can also do different things depending on the state of the system executing the algorithm. Thus, some algorithms, like the one we generally use for long division, contain conditional clauses: if the machine is in such-and-such a state, and receives such-and-so input, then it does this-and-that and moves into this other state.

Computers merely follow algorithms. Moreover, every step of the algorithm can be specified syntactically, by its inscription. When we play a video game, we see cars and people, and hear music. We interact with the machine on a semantic level. But, the computer is just processing syntax, crunching 0s and 1s. So, if strong AI and functionalism are right, then human behavior must be describable algorithmically as well, and representable in purely syntactic form, using a formal language like the one we use in logic.

Searle's Chinese room example is closely related to the qualia objections to functionalism. Searle provides an example of a person working according to purely formal, syntactic rules.

> Imagine that a bunch of computer programmers have written a program that will enable a computer to simulate the understanding of Chinese. So, for example, if the computer is given a question in Chinese, it will match the question against its memory, or data base, and produce appropriate answers to the questions in Chinese. Suppose for the sake of argument that the computer's answers are as good as those of a native Chinese speaker. Now then, does the computer, on the basis of this, understand Chinese, does it literally understad Chinese in the way that Chinese speakers understand Chinese? Well, imagine that you are locked in a room, and in this room are several baskets full of Chinese symbols. Imagine that you (like me) do not understand a word of Chinese, but that you are given a rule book in English for manipulating these Chinese symbols. The rules specify the manipulations of the symbols purely formally, in terms of their syntax, not their semantics. So the rule might say: 'Take a squiggle-squiggle sign out of basket number one and put it next to a squoggle-squoggle sign from basket number two.' Now suppose that some other Chinese symbols are passed into the room, and that you are given further rules for passing back Chinese symbols out of the room. Suppose that unknown to you the symbols passed into the room are called 'questions' by the people outside the room, and the symbols you pass back out of the room are called 'answers to the questions.' Suppose, furthermore, that the programmers are so good at designing the programs and that you are so

> good at manipulating the symbols, that very soon your answers are indistinguishable from those of a native Chinese speaker. There you are locked in your room shuffling your Chinese symbols and passing out Chinese symbols in response to incoming Chinese symbols (Searle, "Can Computers Think?" 671).

You, in the Chinese room, have all the same input as a speaker of Chinese. You produce the same output. But you lack any understanding of Chinese, and there is no way for you to acquire that knowledge by merely manipulating formal symbols. Even if you internalize all the formal rules, the you lack any understanding about the content of the symbols you are manipulating.

Searle extends the argument to robots. Even if they are highly complex, essentially they are doing the same thing that they would be doing if I were controlling the robot from the Chinese room. Any syntactic processor, completely describable in terms of formal processing, is necessarily not a mind.

4.4.5. Searle's Argument

We can present Searle's argument against functionalism and AI as follows.

1. Computer programs are entirely defined by their formal, syntactic structure.
2. Minds have semantic contents.
3. Syntax is not sufficient for semantics.

C: Computer programs are not sufficient for minds.

Premise 1 is obvious, even definitional. Premise 2 is also uncontroversial. We all have minds and we all process meanings. The role of the Chinese-room example is to support premise 2.

Regarding AI, the importance of Searle's argument is that a mechanical model of the mind could not *be* a mind. Any artefact would have to have the causal powers of the mind in order to be a mind. Syntax alone seems insufficient. Though, if our reasoning proceeds according to rules of formal logic, then it would seem that we can have a purely syntactic description of our mental lives.

What is it about our brains, and perhaps our bodies, that allows us to understand, as well as process, information? Searle thinks it has something to do with the way our bodies are connected to the world. He insists that the brain, and its causal connections with sensory organs, and the rest of the body, is essential for understanding our minds. In other words, consciousness is essentially a biological phenomenon. If so, then perhaps the chauvinism of identity theory was right after all.

Paper Topics

1. Is there artificial intelligence? How might the defender of strong AI respond to Searle's criticisms? See Dennett, especially.

2. What is a mind? Compare and contrast two or three theories of the mind. See Churchland's first chapter, and consider Searle's argument.

3. Is logic purely syntactic? Consider Frege's microscope analogy, from the preface to the *Begriffsschrift*, and the discussions of semantics from later in the term.

Suggested Readings

Block, Ned. "Troubles with Functionalism." In *Readings in the Philosophy of Psychology*, vol. 1, Ned Block ed., Harvard University Press, 1980. The source of the Chinese Nation thought experiment, and a sophisticated, detailed examination of the theory of mind most closely associated with artificial intelligence.

Boolos, George, John Burgess and Richard Jeffrey. *Computability and Logic*, 5th ed. Cambridge University Press, 2007. A great technical book on the logic of computation.

Churchland, Paul M. *Matter and Consciousness*, revised edition. MIT Press, 1988. The first few chapters are an excellent overview of the most important theories of the mind.

Dennett, Daniel. *Brainstorms: Philosophical Essays on Mind and Psychology*. MIT Press, 1990. See especially the first essay, "Intentional Systems," for a philosophical defense of AI.

Frege, Gottlob. *Begriffsschrift*. In van Heijenoort. The original.

Searle, John. "Can Computers Think?" In *Philosophy of Mind: Classical and Contemporary Readings*, David Chalmers ed., Oxford University Pres, 2002.

Searle, John. "Minds, Brains, and Programs." In *The Nature of Mind*, David Rosenthal ed., Oxford University Press, 1991. Includes responses and counter-responses from Jerry Fodor and Searle.

§5: Adequacy

4.5.1. Choosing Operators for Formal Languages

There are many different possible logical languages and formal systems. For propositional logic, we can choose a different notation for variables. **PL** has only 26 variables, but we can devise methods to construct indefinitely many variables by indexing.

P, P', P'', P'''...
P_1, P_2, P_3, P_4,...

We can also choose different logical operators. **PL** uses one unary operator, ~, and four binary operators: •, ∨, ⊃, and ≡. There are only four possible unary operators in a bivalent logic (i.e. a logic with two truth values). Given one variable, and a truth table of two rows, there are only four possible distributions of truth values. Any unary operator will have to produce one of the four tables U1 - U4

U1

	α
1	1
1	0

U2

	α
1	1
0	0

U3

~	α
0	1
1	0

U4

	α
0	1
0	0

Only the negation, U3, is useful and has a common name. U2 just repeats the value of the given formula, and so is otiose. We could call U1 a truth operator, since it takes the value '1' whatever the value of α. If we want a formula that produces a truth no matter what the values of the component variables, we can just use any tautology, like EM.

EM P ∨ ~P

Similarly, U4 is a falsity operator, always giving the value '0'. If we want a formula to produce falsity, we can use any contradiction, like PC.

PC P • ~P

There are sixteen possible combinations of truth values for binary operators, sixteen possible truth tables. The following table presents all sixteen possible combinations of truth values given two propositions, α and β.

α	**β**		∨		⊃				≡				•				
1	**1**	1	1	1	1	0	1	1	1	0	0	0	1	0	0	0	0
1	**0**	1	1	1	0	1	1	0	0	1	1	0	0	1	0	0	0
0	**1**	1	1	0	1	1	0	1	0	1	0	1	0	0	1	0	0
0	**0**	1	0	1	1	1	0	0	1	0	1	1	0	0	0	1	0

Notice that the first column after the variables has '1' in all four rows; the next four columns have three '1's and one '0'; the next six columns have all possible distributions of two '1's and two '0's; four columns of three '0's and one '1' follow; lastly, there is the single column of all '0's. There are no other possible distributions of '1's and '0's in a four-row truth table.

Only four of the possible sixteen possible truth tables have names in **PL**. We could give names to others and include them in our language. But, the more operators we include, the more truth tables we have to remember. A language can get clunky and awkward with too many elements. Furthermore, when we want to prove theorems about our formal language, it is useful to have as few elements of the language as possible.

In the other direction, we might want to reduce the vocabulary of our language as far as possible in order to make our metalinguistic proofs easier. But, we want to ensure that the expressive capacity of our language is not reduced. It's fine to remove some of the vocabulary as long as there are other ways of saying what we want to say. To use an analogy from natural language, we could ban the word 'bachelor' from English as long as we still had the words 'unmarried' and 'man'. But, if we got rid of all ways of saying that some man is unmarried, then we would not be able to express some propositions.

When designing logical languages and formal systems, then, we have to balance the ease of using the language with the ease of constructing metalinguistic proofs about the language. We want to make sure both that the language is managable and that it allows us to say what we want to say.

4.5.2. Eliminating the Biconditional and Conditional

One natural way to reduce the vocabulary of our language is to eliminate operators, as long as we can construct statements with the same truth values that those operators produced. We have seen how to eliminate the biconditional by defining it in terms of the conditional. This was the rule we called material equivalence. Call a connective *superfluous* if it can be defined in terms of other connectives.

T1 The biconditional is superfluous.

To prove T1, we just need to show that '$\alpha \equiv \beta$' and '$(\alpha \supset \beta) \bullet (\beta \supset \alpha)$' are logically equivalent. We can do this by method of truth tables.

α	≡	β
1	**1**	1
1	**0**	0
0	**0**	1
0	**1**	0

(α	⊃	β)	•	(β	⊃	α)
1	1	1	**1**	1	1	1
1	0	0	**0**	0	1	1
0	1	1	**0**	1	0	0
0	1	0	**1**	0	1	0

Notice that the conditional is superfluous also, according to the rule of material implication.

T2 The conditional is superfluous.

We can prove T2 by the method of truth tables, just as we did for T1.

α	⊃	β
1	**1**	1
1	**0**	0
0	**1**	1
0	**1**	0

~	α	∨	β
0	1	**1**	1
0	1	**0**	0
1	0	**1**	1
1	0	**1**	0

An alternate way of proving T2 uses metalinguistic versions of conditional and indirect proof. To show that two statements are logically equivalent, metalinguistically, we show that each entails the other. For T2, first we assume: '$\alpha \supset \beta$' is true and show that the truth of '$\sim\alpha \vee \beta$' follows. Assume not. Then some formula of the form '$\sim\alpha \vee \beta$' is false. Then the formula replacing α will have to be true (to make $\sim\alpha$ false) and the formula replacing β will have to be false. But, those values will make '$\alpha \supset \beta$' false, contradicting our assumption.

Next. we assume '$\sim\alpha \vee \beta$' is true and show that the truth of '$\alpha \supset \beta$' follows. Assume that some formula of the form '$\alpha \supset \beta$' is false. Then the value of the formula replacing α must be true and the value of the formula replacing β must be false. But, on those values, '$\sim\alpha \vee \beta$' is false, again contradicting our assumption. QED.

We can go back to prove T1 by the same metalinguistic method. First we assume '$\alpha \equiv \beta$' and show that '$(\alpha \supset \beta) \bullet (\beta \supset \alpha)$' follows. Then, we assume '$(\alpha \supset \beta) \bullet (\beta \supset \alpha)$' and show that '$\alpha \equiv \beta$' follows. I leave the details to the reader.

Both of these methods for proving T1 and T2 produce the same results, since they depend on the same truth values. A third method of proving the equivalence of two statements is to derive one from the other using the rules of inference we introduced in Chapter 2. For T2, for example, we assume: '$\alpha \supset \beta$ and derive '$\sim\alpha \vee \beta$'; then we assume '$\sim\alpha \vee \beta$' and derive '$\alpha \supset \beta$'. In this case, our proof will use metalinguistic formulas rather than formulas of **PL**, but the same rules apply.

So, as a methodological observation, we have on our hands two distinct notions of logical equivalence.

LE_1 Two statements are logically equivalent iff they have the same values in every row of the truth table.

LE_2 Two statements are logically equivalent iff each is derivable from the other.

We hope that LE_1 and LE_2 yield the same results. In order to show that this is the case, we must show that our formal system is sound and complete, which is a topic for another section.

Combining T1 and T2, we discover that any sentence which can be written as a biconditional can be written in terms of negation, conjunction, and disjunction. To eliminate the biconditional and the conditional from a sentence like DB which is naturally translated using the ≡, we use two steps. After regimenting DB directly as a biconditional, we eliminate the biconditional and then get rid of the conditional.

DB Dogs bite if and only if they are startled.
$B \equiv S$
$(B \supset S) \bullet (S \supset B)$
$({\sim}B \vee S) \bullet ({\sim}S \vee B)$

Two questions arise from these considerations about eliminating operators. First, how can we be sure that all sentences can be written with just the five (or, now, three) connectives of **PL**? Second, can we eliminate more connectives? What is the fewest number of connectives that we need? We will answer both questions in the remainder of this section.

4.5.3. Defining Adequacy and Disjunctive Normal Form

A set of connectives is called **adequate** if and only if corresponding to every possible truth table there is at least one sentence using only those connectives. By "every possible truth table," I mean every combination of '1's and '0's in the column under the main operator. We want our connectives to be adequate so that we can construct formulas with all possible truth conditions. If our set of connectives is adequate, then our propositional logic will be able to say anything that any propositional logic can say.

To give you a taste of what we are after, consider a severely limited adequacy result.

T3 Negation and conjunction are adequate, if we use only one propositional variable.

We can prove T3 by sheer force. There are only four possible truth tables: 11, 10, 01, 00. Here are statements for each of them which use no connectives other than negation and conjunction.

~	(α	•	~	α)
1	1	0	0	1
1	0	0	1	0

α
1
0

~	α
0	1
1	0

α	•	~	α
1	**0**	0	1
0	**0**	1	0

We want to demonstrate the general theorem that the five connectives of **PL** are adequate for any number of propositional variables. By T1 and T2, we know that the five connectives are adequate if, and only if, the three (negation, conjunction, and disjunction) are adequate.

In order to prove the general theorem, consider the set of wffs of **PL** that are in *Disjunctive Normal Form* (DNF). A sentence is in DNF if it is a series of disjunctions, each disjunct of which is a single letter, a negation of single letters, or a conjunction of simple letters or negations of simple letters. A single letter or its negation can be considered a degenerate conjunction or disjunction. (For the purposes of this section, we can drop brackets among three or more conjuncts, or three or more disjuncts, though we still need brackets when conjoining disjunctions or disjoining conjunctions.) DNF lists some sentences in DNF. NDNF lists some sentences that are not in DNF.

DNF	P
	P • ~Q
	~P ∨ Q
	(P • Q) ∨ (~P • Q)
	~P ∨ ~Q ∨ (~P • ~Q)
NDNF	~(P • Q)
	P ⊃ Q
	(P • ~Q) ∨ (~P ≡ Q)
	(P ∨ Q) • (~P ∨ ~Q)
	P ∨ ~Q ∨ ~(P ∨ Q)

Notice that the first and last sentences in NDNF are logically equivalent to related sentences in DNF. '~(P • Q)' is logically equivalent to '~P ∨ ~Q'. 'P ∨ ~Q ∨ ~(P ∨ Q)' is logically equivalent to 'P ∨ ~Q ∨ (~P • ~Q)'. These equivalences are easily shown by constructing the appropriate truth tables.

4.5.4. Proving Adequacy and Inadequacy for Familiar Connectives

We have seen that the set of five connectives of **PL** is adequate if the set of three {~, •, ∨} is. The proof of T4 will show that both sets are indeed adequate.

T4 The set of negation, conjunction, and disjunction {~, •, ∨} is adequate.

The proof of T4 proceeds by cases. We will see a way to construct a sentence using only the three connectives for any possibility of combinations of truth values in any truth table.

For any size truth table, with any number of connectives, there are three possibilities for the column under the main operator.

Case 1: Every row is false.
Case 2: There is one row which is true, and every other row is false.
Case 3: There is more than one row which is true. (Perhaps even all the rows are true.)

For Case 1, we can construct a sentence with one variable in the sentence conjoined with its negation and each of the remaining variables. So, if you have variables P, Q, R, S, and T, you would write, 'P • ~P • Q • S • T'. If you have more variables, add more conjuncts. The resulting formula, in DNF, is false in every row, since each row contains a contradiction. It uses only conjunction and negation.

For Case 2, consider the row in which the statement is true. We can write a conjunction of the following statements:

For each variable, if it is true in that row, write that variable.
For each variable, if it is false in that row, write the negation of that variable.

The resulting formula is in DNF (the degenerate disjunction) and is true in only the prescribed row. For example, consider a formula with two variables, P and Q, and the column under the main operator. The formula may be as complicated as we wish. We could be considering '~(~P ∨ ~~Q)' or '~[(P ∨ P) ⊃ (Q • ~~Q)]', each of which yields the given truth table. We are concerned to construct a formula, in DNF, which matches the single column under the main operator in any such formula of **PL**.

P	Q	Main operator
1	1	0
1	0	1
0	1	0
0	0	0

We consider the second row only, in which P is true and Q is false. Our conjunction will be 'P • ~Q'. This formula is in DNF and it is logically equivalent, by defnition, to whatever the original sentence was, no matter which of the five connectives it used.

Also, notice that our sentence in DNF is equivalent to a different statement, '~(~P ∨ Q)'; you can see their equivalence by constructing the truth table. Many different formulas which will yield the same truth table. In fact, there are infinitely many ways to produce each truth table. For example, one can always just add pairs of ~s to a formula.

For Case 3, we just repeat the method from Case 2 for each row in which the statement is true. Then, we form the disjunction of all the resulting formulas. Again, the resulting formula will be in DNF and be logically equivalent to the original formula no matter which connectives it used. Here is an example.

P	Q	R	Main operator
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0

To construct a formula with that truth table, we need to consider only the first and fourth rows. In the first row, all variables are true. In the fourth, 'P' is true, but 'Q' and 'R' are false. Our resultant formula will be '(P • Q • R) ∨ (P • ~Q • ~R)'. Punctuation can easily be added to make the formula well-formed. QED.

Given T4 and the methods used in the proofs of Theorems 1 and 2, we can easily prove several other sets of connectives adequate.

T5 The set {∨, ~} is adequate.

To prove T5, we use can use the method in T4 to write a formula for any truth table using as connectives only those in the set {∨, •, ~}. But any statement of the form 'α • β' is equivalent to one of the form '~(~α ∨ ~β)'. So, we can replace any occurrence of '•' in any formula, according to the above equivalence.

The proofs of T6 and T7 are just as straightforward. For T6, we merely require a formula uusing conjunction and negation which is logically equivalent to 'α ∨ β'. T7 is a little trickier. I leave the proofs of both to the reader.

T6 The set {•, ~} is adequate.
T7 The set {~, ⊃} is adequate.

We have seen that some pairs of connectives are adequate to express any truth table. But, not all sets of pairs of connectives are adequate.

T8 The set {⊃, ∨} is inadequate.

To show that a set of connectives is inadequate, we can show that there is some truth table that can not be constructed using those connectives. Recall that both 'α ⊃ β' and 'α ∨ β' are true when α and β are both true. Thus, using these connectives we can never construct a truth table with a false first row. {⊃, ∨} is inadequate.

All of the sets of single connectives in **PL** are inadequate. T9 is an example.

T9 The set {⊃} is inadequate.

To prove T9, consider the truth table for conjunction. We want to construct a formula, using ⊃ as the only connective, which yields the same truth table. Imagine that we have such a formula, and imagine the smallest such formula. Since, the only way to get a 0 with ⊃ is with a false consequent, the truth table of the consequent of our formula must either be 1000 or 0000. Since we are imagining that our formula is the smallest formula which yields 1000, the consequent of our formula must be a contradiction. But, the only way to get a contradiction, using ⊃ alone, is to have one already! Since we can not construct the contradiction, we can not construct the conjunction.

We will need one more inadequate set for the proof of T13.

T10 The set {~} is inadequate.

To prove T10, we need only one variable. The only possible truth tables with one variable and ~ are 10 and 01. Thus, we can not generate 11 or 00.

4.5.5. Adequacy for New Connectives

Despite the inadequace of our single connectives, there are sets of single connectives which are adequate. Consider the Sheffer stroke, '|', which is also called alternative denial, or not-both.

α	|	β
1	0	1
1	1	0
0	1	1
0	1	0

T11 The set {|} is adequate.

To prove T11, notice that '~α' is logically equivalent to 'α | α' and that 'α • β' is logically equivalent to '(α | β) | (α | β)'. By T6, {~, •} is adequate. So, T11 follows.

There is one more adequate single-membered set. Consider the connective for the Peirce arrow, '↓', also called joint denial, or neither-nor.

α	↓	β
1	0	1
1	0	0
0	0	1
0	1	0

T12 The set {↓} is adequate.

The proof of T12 uses T5 and the equivalence of '~α' to 'α ↓ α' and of 'α ∨ β' to '(α ↓ β) ↓ (α ↓ β)'.

Both | and ↓ were initially explored by C.S. Peirce, though Henry Sheffer gets his name attached to the former for his independent work on it. Given that both the Sheffer stroke and the Peirce arrow are adequate, we could build systems of propositional logic around just those connectives. Translations between such logical languages and English would be difficult, and our propositions would get complex quickly. We have to balance the virtues of having fewer connectives with the virtues of languages with which it is easier to work. We could easily add either the Sheffer stroke or the Peirce arrow to **PL**; they would be superfluous just like the material biconditional.

4.5.6. The Limit of Adequacy

Lastly, we can prove that there are no other single, adequate connectives.

T13: ↓ and | are the only connectives which are adequate by themselves.

To prove T13, imagine we had another adequate connective, #. We know the first rows must be false, by the reasoning in the proof of T8. Similar reasoning fills in the last row.

α	#	β
1	0	1
1		0
0		1
0	1	0

Thus, '~α' is equivalent to 'α # α'. Now, we need to fill in the other rows. If the remaining two rows are 11, then we have '|'. If the remaining two rows are 00, then we have '↓'. So, the only other possibilities are 10 and 01. 01 yields 0011, which is just '~α'. 10 yields 0101, which is just '~β'. By T10, {~} is inadequate. QED

Exercises A. Which of the following sentences are in DNF?

1. (P • ~Q) ∨ (P • Q)
2. (P • Q • R) ∨ (~P • ~Q • ~R)
3. ~P ∨ Q ∨ R
4. (P ∨ Q) • (P ∨ ~R)
5. (P • Q) ∨ (P • ~Q) ∨ (~P • Q) ∨ (~P • ~R)
6. (~P • Q) • (P • R) ∨ (Q • ~R)
7. (P • ~Q • R) ∨ (Q • ~R) ∨ ~Q
8. ~(P • Q) ∨ (P • R)
9. P • Q
10. ~P

Exercises B

1. Use the metalinguistic, semantic form which I used to prove T2 to prove T1.
2. Prove T6.
3. Prove T7.

Solutions to Exercises A

Only 4, 6, and 8 are not in DNF, though 8 could be quickly put into DNF

Paper Topics

1. While there are no other adequate single sets of connectives, there are other binary connectives. Are there other adequate pairs? If so, which? If not, why not?

2. What are the meanings of the other possible binary connectives? Can a good argument be made to use any others in translation from natural language into a formal language?

3. Why are there only unary and binary connectives?

4. Try to construct a formal language with the same expressive powers as **PL** but with with none of the standard connectives.

VIII. Suggested Reading

Susan Haack, *Philosophy of Logics*, Chapter 3, has a discussion of adequacy.

Geoffrey Hunter, *Metalogic*. The results above are mostly contained in §21. The references below are mostly found there, as well. His notation is a bit less friendly, but the book is wonderful, and could be the source of lots of papers.

Elliott Mendelson, *Introduction to Mathematical Logic*. Mendelson discusses adequacy in §1.3. His notation is less friendly than Hunter's, but the exercises lead you through some powerful results.

Emil Post, "Introduction to a General Theory of Elementary Propositions", reprinted in van Heijenhoort. The notation is different, but the concepts are not too difficult. It would be interesting to translate into a current notation, and present some of the results.

Several papers from C.S. Peirce initially explored the single adequate connectives. They might be fun to work through. I can give you references.

§6: Three-Valued Logics

4.6.1. Eight Motivations for Three-Valued Logics

We have been both working with our system of propositional logic, in the object language, and interpreting our system, doing semantics in a metalanguage. The semantics for **PL** mainly consists in the assignments of truth values to propositional variables. We represent these assignments using the truth tables. For the most part, in this book, we use a bivalent interpretation of our logic. A bivalent semantics is one with just two truth values: 1 and 0.

For various reasons, some logicians have constructed semantics for propositional logic with more than two truth values. These semantics divide into two classes. The first type of non-bivalent semantics uses three truth values. The second type uses more than three truth values. Most of the interpretations which use more than three truth values use infinitely many truth values. We can take '0' for absolute falsity, '1' for absolute truth, and all real numbers between 0 and 1 as different truth values landing somewhere between absolute truth and absolute falsity. Such semantics are called fuzzy logics.

Three-valued interpretations are generally called three-valued logics. That name is infelicitous, since the difference between bivalent and three-valued interpretations comes not in the object-language logic but in the metalanguage semantics. Still, I will use the term 'three-valued logic' for consistency with common usage. In this section, we will look at eight motivations, M1-M8, for three-valued logics. Some of these topics are examined in depth elsewhere in this text and so the discussion here will be brief.

M1. Mathematical sentences with unknown truth values
M2. Statements about the future
M3. Failure of presupposition
M4. Nonsense
M5. Programming needs
M6. Semantic paradoxes
M7. The paradoxes of the material conditional
M8. Vagueness

M1. Mathematical sentences with unknown truth values

Some philosophers introduce a third truth value to classify sentences whose truth values are unknown to us. In particular, mathematical sentences like GC seem puzzling.

GC Every even number greater than four can be written as the sum of two odd primes.

GC is called Goldbach's conjecture, though Euler actually formulated it in response to a weaker hypothesis raised by Goldbach in 1742. Goldbach's conjecture has neither been proved true nor disproved. It has been verified up to very large values. There are websites at which you can test any number. As of this writing, there is a computer working on larger and larger numbers to verify that GC holds; it has passed 10^{18}.

There are, also, inductive arguments which make mathematicians confident that Goldbach's conjecture is true. As the numbers grow, the number of different pairs of primes that sum to a given number (called Goldbach partitions) tends to grow. Even for the relatively small numbers between 90,000 and 100,000, there are no even numbers with fewer than 500 Goldbach partitions. It would be extremely surprising if the number suddenly dropped to 0. Still, many smart mathematicians have tried and failed to devise a proof of GC.

We might take Goldbach's conjecture to be neither true nor false. We might do so, especially, if we think that mathematics is constructed, rather than discovered. If and when some one proves it, or its negation, then we could apply a truth value to the proposition. Until we have a proof, we could take Goldbach's conjecture, and other unproven mathematical claims, to lack a truth value.

M2. Statements about the future

Mathematical sentences like GC are alluring for those who favor three-valued logics. But, many mathematicians believe that GC is either true or false. The reason we can not decide whether it is true or false is that we have limited intelligence. But, the sentence has a definite truth value. Such sentiments are not as strong in cases like PT.

PT There will be a party tomorrow night at Bundy Dining Hall.

Maybe there will be a party in Bundy tomorrow; maybe there will not be. Right now, we can not assign a truth value to PT.

The classic discussion of the problem in PT, generally labeled the problem of future contingents, may be found in Aristotle's *De Interpretatione* regarding a sea battle. Since PT is contingent, we can, at this moment, neither assert its truth nor its falsity.

> In things that are not always actual there is the possibility of being and of not being; here both possibilities are open, both being and not being, and consequently, both coming to be and not coming to be (*De Interpretatione* §9.19a9-13).

We know that one of the two truth values will apply, eventually. Either there will be a party tomorrow night or there will not be. Right now, though, PT seems to lack a truth value.

> It is necessary for there to be or not to be a sea-battle tomorrow; but it is not necessary for a sea-battle to take place tomorrow, nor for one not to take place - though it is necessary for one to take place or not to take place (*De Interpretatione* §9.19a30-33).

If the claim that there will be a sea-battle tomorrow has a truth value now, then the event is not contingent; it is already determined. Since the future is not determined, the truth values of statements about the future should also be undetermined.

We can understand Aristotle's claim better by considering the following three claims.

EM Either there will be a sea-battle tomorrow or there will not be a sea-battle tomorrow.
EM1 There will be a sea-battle tomorrow.
EM2 There will not be a sea-battle tomorrow.

Aristotle wants to call EM true, indeed necessarily true, while withholding truth values from EM1 and EM2. If EM1 and EM2 are not true, and we only have two truth values, then they must be false. If EM1 and EM2 are false, we should be willing to assert their negations.

EM1' It is not the case that there will be a sea-battle tomorrow.
EM2' It is not the case that there will not be a sea-battle tomorrow.

EM1' and EM2' represent our acknowledgment of the contingency of the event. But, taken together, EM1', and EM2' form a contradiction.

EMC $\sim P \bullet \sim\sim P$

~	P	•	~	~	P
0	1	**0**	1	0	1
1	0	**0**	0	1	0

If we have a third truth-value, we can assert both EM1' and EM2' without contradiction. In a three-valued logic, denying that a statement is true does not entail that it is false. It can be neither true nor false.

M3. Failure of presupposition

Neither KB nor KN are true propositions.

KB The king of America is bald.
KN The king of America is not bald.

But, KN looks like the negation of KB. If we regiment KB as ‘P’, we should regiment KN as ‘~P’. In a bivalent logic, since ‘P’ is not true, we must call it false, since we have only two truth values. Assigning the value ‘false’ to ‘P’ means that ‘~P’ should be assigned ‘true’. But, KN is false.

The problem is that in this case we want both a proposition and its negation to be false. But in a bivalent logic, the negation of a false proposition is a true proposition. Thus, we can never, in a bivalent logic, deny both a statement and its negation, as we wish to do with KB and KN.

We think that KB and KN are both false because they both contain a false presupposition. FP1 - FP3 all contain a failure of presupposition. FP3 contains a failure of presupposition even though it is not declarative.

FP1 The woman on the moon is six feet tall.
FP2 The rational square root of three is less than two.
FP3 When did you stop beating your wife?

One response to the problem of presupposition failure in propositions is to call such propositions neither true nor false.

M4. Nonsense

The distinction between syntax and semantics can be a motivation to adopt a third truth-value. The syntax of a formal language tells us whether a string of symbols of the language is a wff. The correlate of syntax, in natural language, is grammaticality. But, not all grammatical sentences are sensible. We might consider some grammatical but nonsensical sentences to lack truth-values.

GN1 Quadruplicity drinks procrastination. (From Bertrand Russell)
GN2 Colorless green ideas sleep furiously. (From Noam Chomsky)

In the syntax of English, GN1 and GN2 are well-formed. But, their well-formedness does not entail that we can assign truth values to them. If we adopt a three-valued logic, we can assign them the missing truth value and save falsity for sentences that are sensible as well as gramamtical.

M5. Programming needs

Logic is essential to the program of a computer. At the most basic level, computer circuits are just series of switches which can either be open or closed. To represent the circuit, we use logic. Roughly, electrons can pass through a switch if it is closed and can not pass through if it is open. To interpret the state of the circuit, we take a closed switch to be true and an open switch to be false. We might want to leave the values of some variables unassigned, during a process. For example, we might want to know how a system works without knowing whether a given switch is open or closed. A three-valued logic can thus model closed switches (true), open switches (false), and switches which we do not know whether they are open or closed (undetermined).

M5 is merely a pragmatic motivation, rather than a philosophical one, for developing three-valued logics. It is, though, a reason to explore the merits of such semantics. The development of a three-valued semantics is a formal, technical project. We can wonder about the philosophical motivations for and applications of our logic and its semantics independently of their construction.

M6. Semantic paradoxes

There are a variety of semantic paradoxes. The most famous is called the liar.

L L is false.

L is an example of a paradoxical sentence. If L is true, then L is false, which makes L true, which makes L false... L seems to lack a definite truth value, even though it is a perfectly well-formed sentence.

L is often called Epimenides' paradox. Epimenides was a Cretan to whom the statement that all Cretans are liars is attributed. Since assigning 1 or 0 to L leads to a contradiction, we might assign it a third truth value.

M7. The paradoxes of the material conditional

Statements of the form MC1 - MC3 are sometimes called paradoxes of the material conditional. We call them paradoxes, a name which is probably too strong, beacuse they turn out to be logical truths even though they are not obvious.

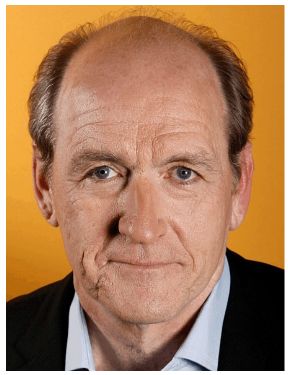
MC1 $\alpha \supset (\beta \supset \alpha)$
MC2 $\sim\alpha \supset (\alpha \supset \beta)$
MC3 $(\alpha \supset \beta) \vee (\beta \supset \alpha)$

The paradoxes of the material conditional have awkward consequences. MC1 says, approximately, that if a statement is true, then anything implies it. For, the truth table for the material conditional is true on every line in which the consequent is true. MC2 says that if a statement is false, its opposite entails any other statement. MC3 says that for any statement, β, either any other statement entails it, or it entails any statement. Every statement must be either true or false. If a given statement is true, then any statement entails it. If a given statement is false, then it entails any statement.

On the one hand, MC1 - MC3 are logical truths. We certainly can not call their instances false. On the other hand, their instances are not the kinds of sentences that some people feel comfortable calling true. So, we might use a third truth value for them.

M8. Vagueness

As a last motivation for three-valued logics, consider the phenomenon of vagueness. Many predicates admit of borderline, or vague, cases. For example, consider baldness. There are paradigm instances of baldness which are incontrovertible. There are also paradigm instances of non-baldness. But, there are also cases in which we don't know what to say.


V1	Tyra Banks is bald.
V2	Richard Jenkins is bald.
V3	Devendra Banhart is bald.

V1 is true; V3 is false. But between Tyra Banks and Devendra Banhart, there is a penumbra. We could, if we wish, give V2 a third truth value. More compellingly, we could apply a logic in which there are infinitely many truth values to this case. To do so, we could assign a value of 0 to V3, and a value of 1 to V1. Then, we can assign any real-number between 0 and 1, say .3, to V2.

4.6.2 Three-Valued Logics

The rules for determining truth values of formulas in a logic are called the semantics. We provided semantics for propositional logic by constructing truth tables. Since we used only two values, true and false, our semantics is called two-valued. Our two-valued semantics is also called classical semantics If we want to adopt a third truth value, which we might call unknown, or indeterminate, we must revise all the truth tables. I will call the third truth value indeterminate, and use the Greek letter ι to indicate it. Remember, the idea is that we can ascribe ι to sentences which lack a clear truth value.

There are two options for how to deal with unknown or indeterminate truth values in the new semantics. First, one could claim that any indeterminacy among component propositions creates indeterminacy in the whole. This the principle underlying Bochvar's semantics, which is sometimes called Weak Kleene semantics. Second, one could try to ascribe truth values to as many formulas as possible, despite the indeterminate truth values. For example, a conjunction with one false conjunct could be ascribed falsity whether the other conjunct is true, false, or unknown. This is the principle underlying Strong Kleene semantics and Lukasiewicz semantics.

We proceed to look at these three different three-valued semantics. We will look at: 1. The rules for each; 2. How the new rules affect the logical truths (tautologies); and 3. How the new rules affect the allowable inferences (valid arguments). To show the semantics, I will present truth tables for the standard connectives. For simplicity, I will ignore the biconditional.

Bochvar semantics (B)

α	~α
1	0
ι	ι
0	1

α	•	β
1	1	1
1	ι	ι
1	0	0
ι	ι	1
ι	ι	ι
ι	ι	0
0	0	1
0	ι	ι
0	0	0

α	∨	β
1	1	1
1	ι	ι
1	1	0
ι	ι	1
ι	ι	ι
ι	ι	0
0	1	1
0	ι	ι
0	0	0

α	⊃	β
1	1	1
1	ι	ι
1	0	0
ι	ι	1
ι	ι	ι
ι	ι	0
0	1	1
0	ι	ι
0	1	0

In Bochvar semantics, no classical tautologies come out as tautologies. Consider, under Bochvar, ‘P ⊃ P’ and ‘P ⊃ (Q ⊃ P)’.

P	⊃	P
1	1	1
ι	ι	ι
0	1	0

P	⊃	(Q	⊃	P)
1	1	1	1	1
1	ι	ι	ι	1
1	1	0	1	1
ι	ι	1	ι	ι
ι	ι	ι	ι	ι
ι	ι	0	ι	ι
0	1	1	0	0
0	ι	ι	ι	0
0	1	0	1	0

These two classical tautologies, and all others, do not come out false on any line on Bochvar semantics. But, they do not come out as true on every line. This result is generally undesirable, since the classical tautologies seem pretty solid. Tautologies are also known as logical truths. They are the theorems of the logic.

For those motivated by the paradoxes of the material conditional, Bochvar semantics could be tempting. Other systems of logic, called relevance logics, attempt to keep most classical logical truths, but eliminate the paradoxes of material implication. Unfortunately, Bochvar semantics seems too strong a reaction to the oddities of those so-called paradoxes; it eliminates all classical tautologies.

One solution to the problem of losing logical truths in Bochvar semantics would be to redefine ‘tautology’ as a statement which never comes out as false. Redefining ‘tautology’ in this way, though, weakens the concept, making it less useful

Next, consider what Bochvar semantics does to validity. We defined a valid argument as one for which there is no row in which the premises are true and the conclusion is false. We could have defined a valid argument as one for which there is no row in which the premises are true and the conclusion is not true. Classically, these two definitions are equivalent. But, in three-valued semantics, they cleave. If we take a row in which the premises are true and the conclusion is indeterminate as a counterexample to an argument, as Bochvar did, then some classically valid inferences come out invalid.

Under classical semantics, ‘P / Q ∨ P’ is a valid inference.

P	//	Q	∨	P
1		1	**1**	1
1		0	**1**	1
0		1	**1**	0
0		0	**0**	0

Under Bochvar semantics, the argument comes out invalid. The second row is a counterexample.

P	//	Q	∨	P
1		1	**1**	1
1		ι	**ι**	1
1		0	**1**	1
ι		1	**ι**	ι
ι		ι	**ι**	ι
ι		0	**ι**	ι
0		1	**1**	0
0		ι	**ι**	0
0		0	**0**	0

Bochvar semantics proceeds on the presupposition that any indeterminacy infects the whole. It thus leaves the truth values of many formulas undetermined. But, we might be able to fill in some of the holes. That is, why should we consider the disjunction of a true statement with one of indeterminate truth value to be undetermined? Or, why should we consider the conditional with an antecedent of indeterminate truth value to itself be of indeterminate truth value, if the consequent is true? Whatever other value we can assign the variables with unknown truth value, both sentences will turn out to be true.

Kleene's semantics leaves fewer rows unknown.

Kleene semantics (K3)

P	~P
1	0
ι	ι
0	1

P	•	Q
1	1	1
1	ι	ι
1	0	0
ι	ι	1
ι	ι	ι
ι	**0**	0
0	0	1
0	**0**	ι
0	0	0

P	∨	Q
1	1	1
1	**1**	ι
1	1	0
ι	**1**	1
ι	ι	ι
ι	ι	0
0	1	1
0	ι	ι
0	0	0

P	⊃	Q
1	1	1
1	ι	ι
1	0	0
ι	**1**	1
ι	ι	ι
ι	ι	0
0	1	1
0	**1**	ι
0	1	0

Kleene semantics has a certain intuitiveness. But, in order to compare Bochvar to Kleene properly, we should look at the differences on logical truths and inference patterns. Consider the same two tautologies, 'P ⊃ P' and 'P ⊃ (Q ⊃ P)' under Kleene semantics:

P	⊃	P
1	**1**	1
ι	**ι**	ι
0	**1**	0

P	⊃	(Q	⊃	P)
1	**1**	1	1	1
1	**1**	ι	1	1
1	**1**	0	1	1
ι	**ι**	1	ι	ι
ι	**ι**	ι	ι	ι
ι	**1**	0	1	ι
0	**1**	1	0	0
0	**1**	ι	ι	0
0	**1**	0	1	0

While many more of the rows are completed, the statements still do not come out as tautologous, under the classical definition of 'tautology'. Lukasiewicz, who first investigated three-valued logics, tried to preserve the tautologies.

Lukasiewicz semantics (L3)

There is only one difference between Kleene semantics and Lukasiewicz semantics, in the fifth row of the truth table for the conditional.

P	~P
1	0
ι	ι
0	1

P	•	Q
1	1	1
1	ι	ι
1	0	0
ι	ι	1
ι	ι	ι
ι	0	0
0	0	1
0	0	ι
0	0	0

P	∨	Q
1	1	1
1	1	ι
1	1	0
ι	1	1
ι	ι	ι
ι	ι	0
0	1	1
0	ι	ι
0	0	0

P	⊃	Q
1	1	1
1	ι	ι
1	0	0
ι	1	1
ι	**1**	ι
ι	ι	0
0	1	1
0	1	ι
0	1	0

One might wonder how we might justify calling a conditional with indeterminate truth values in both the antecedent and consequent true. For, what if the antecedent turns out true and the consequent turns out false? Put that worry aside, and look at what this one small change does.

P	⊃	P
1	**1**	1
ι	**1**	ι
0	**1**	0

P	⊃	(Q	⊃	P)
1	**1**	1	1	1
1	**1**	ι	1	1
1	**1**	0	1	1
ι	**1**	1	ι	ι
ι	**1**	ι	1	ι
ι	**1**	0	1	ι
0	**1**	1	0	0
0	**1**	ι	ι	0
0	**1**	0	1	0

Voila! L3 retains many of the classical tautologies that B and K3 lost. In fact, L3 does not get all classical tautologies, including the law of excluded middle, ‘P ∨ ~P’.

P	∨	~	P
1	**1**	0	1
ι	**ι**	ι	ι
0	**1**	1	0

While excluded middle still does not come out tautologous, that is a law that some folks would like to abandon, anyway.

The fact that L3 recaptures many classical tautologies is advantageous. It remains to be shown that the change in semantics is warranted. Is it acceptable to call true a conditional whose antecedent and consequent are both of the third truth value, undetermined or unknown or indeterminate? The justification for L3 over K3 seems impossible from the bottom up.

The lesson of Lukasiewicz semantics is that we need not give up classical tautologies, logical truths, to have a three-valued logic. The fewer changes we make to the set of logical truths, the less deviant the logic is. But, the semantics which allows us to retain these logical truths may not be as pretty as we would like.

Lastly, consider the effect on validity of moving from Bochvar to Kleene or Lukasiewicz. Consider again the argument: ‘P / Q ∨ P’.

Bochvar

P	//	Q	∨	P
1		1	**1**	1
1		ι	**ι**	1
1		0	**1**	1
ι		1	**ι**	ι
ι		ι	**ι**	ι
ι		0	**ι**	ι
0		1	**1**	0
0		ι	**ι**	0
0		0	**0**	0

counter-example in row 2

Kleene

P	//	Q	∨	P
1		1	**1**	1
1		ι	**1**	1
1		0	**1**	1
ι		1	**1**	ι
ι		ι	**ι**	ι
ι		0	**ι**	ι
0		1	**1**	0
0		ι	**ι**	0
0		0	**0**	0

valid - no counter-example

Lukasiewicz

P	//	Q	∨	P
1		1	**1**	1
1		ι	**1**	1
1		0	**1**	1
ι		1	**1**	ι
ι		ι	**ι**	ι
ι		0	**ι**	ι
0		1	**1**	0
0		ι	**ι**	0
0		0	**0**	0

valid - no counter-example

Both Kleene and Lukasiewicz semantics thus maintain some of the classical inference patterns which are lost in Bochvar semantics.

4.6.3 Problems with Three-Valued Logics

All three-valued logics abandon some classical tautologies and classically valid inference patterns. This result may be acceptable, depending on one's motivation for adopting a three-valued logic. But, it is not clear that all the problems that motivated three-valued logics can be solved by adopting three-valued logics.

For example, Bochvar hoped that his semantics would solve the problems of the semantic paradoxes. The liar sentence can be given a truth value in Bochvar semantics without paradox. That's good. But consider SL, the strengthened liar.

SL	SL is not true

Suppose SL is true. Then what it says, that SL is not true, must hold. So, SL is not true. It might be false or it might be undetermined. In either case, SL, because it says that SL is not true, turns out to be true. The paradox recurs. Adopting the three-valued logic doesn't get rid of the strengthened liar paradox.

Another worry about three-valued logics concerns the interpretation of the third value. Thinking of it as unknown, for example, seems to involve a conceptual confusion. 'Unknown' might be understood not as a third truth value, but as the lack of a truth value. Instead of filling in such cells in the truth table, we could just leave them blank.

Leaving certain cells of the truth table blank is part of what is called the truth-value gap approach. A truth table some of whose cells are left blank is called a partial valuation: only some truth values of complex propositions are computed on the basis of the truth values of component propositions. Faced with partial valuations, the logician may consider something called a supervaluation. A supervaluation considers the different ways to complete partial valuations and classifies formulas and arguments according to the possibilities for completion. Supervaluations can, in certain cases, recapture some of the missing values in a partial valuation.

One serious worry about three-valued logics is called the change of logic, change of subject argument. This argument comes from Quine. The basic idea of the argument is that in order to disagree with someone, you have to at least agree on what you are disagreeing about. There has to be some common ground on which you can stand, to argue, or else you are not really disagreeing at all.

Consider two terms, and their definitions, which I will stipulate.

Chair$_1$	desk chairs, dining room chairs, and such, but not recliners or bean bag chairs
Chair$_2$	all chair$_1$ objects, and also recliners and bean bag chairs

Now, consider one person, who uses 'chair' as chair$_1$ and another person who uses 'chair' as chair$_2$. Imagine these two people talking about a bean bag chair. Person 1 affirms 'that's a chair', while Person 2 denies that sentence. Since they are using the same term, it looks like they are disagreeing. But they are not really disagreeing about whether the bean bag chair is a chair. They are disagreeing about what 'chair' means. They are both correct in their claims about the bean bag. The bean bag is a chair$_1$ and is not a chair$_2$. What looks like a disagreement is not really a disagreement; the subject has been changed.

Quine presented the change of logic, change of subject argument in response to a proposal, related to the introduction three-valued logics, to allow some contradictions in one's language. The problem with accepting contradictions is that they lead to explosion, the inferential process by which any proposition follows from a contradiction. Those who attempt to embrace contradictions have to find a way to block explosion.

> Perhaps, it is suggested, we can so rig our new logic that it will isolate its contradictions and contain them. My view of this dialogue is that neither party knows what he is talking about. They think they are talking about negation, '~', 'not'' but surely the notation ceased to be recognizable as negation when they took to regarding some conjunctions of the form ['P • ~P'] as true, and stopped regarding such sentences as implying all others. Here, evidently, is the deviant logician's predicament: when he tries to deny the doctrine he only changes the subject (Quine, *Philosophy of Logic* 81).

If we are considering debates over the correct logic, even claims of what it means to affirm or deny a sentence are under discussion. Debates over the correct logic seem to be more like the disagreement between $chair_1$ and $chair_2$. The disputants do not agree on the terms they are using, and so are talking past each other.

Imagine we are linguists, and we are headed to a newly-discovered alien planet. We have to translate a completely new language into English. We start by assuming that the aliens obey the rules of logic. If we were to propose a translation of the alien language on which the aliens often made statements that translated into the form of 'P • ~P', we would not assume that these beings are often contradicting themselves. We would revise our translation to make it more charitable. We take the laws of logic as fundamental. We use them as common ground on which to base our translations. If we hypothesize that the native is asserting a contradiction, we take that to be evidence against our translation rather than evidence against the native's intellectual capacity, for example.

We need logic to serve as a starting point for the translation. We need common ground even to formulate disagreement. If we disagree about the right logic, then we have merely changed the subject.

4.6.4 Avoiding Three-Valued Logics

I introduced three-valued logics in order to respond to some problems which arose with classical logic.

M1. Mathematical sentences with unknown truth values
M2. Statements about the future
M3. Failure of presupposition
M4. Nonsense
M5. Programming needs
M6. Semantic paradoxes
M7. The paradoxes of the material conditional
M8. Vagueness

I mentioned that three-valued logics does not solve the problems of the semantic paradoxes. There are ways for the classical logician to deal with all of these problems, anyway. I will not discuss each of them, here. But, here are a few hints to how to solve them.

M1, concerning sentences with unknown truth values and M2, concerning propositions referring to future events, are related. In both cases, we can blame ourselves, rather than the world, for our not knowing the truth value. Thus, we can say that Goldbach's conjecture is either true or false, but we just do not know which. Similarly, we can say that either there will be a party at Bundy Dining Hall tomorrow, or there will not. We need not ascribe a deep problem to truth values. Such sentences have truth values. We just do not know them.

Proponents of classical logic may deal with problems about time by appealing to a four-dimensional framework. We can take a God's-eye point of view and think of the world as a completed whole, from the beginning until the end of time. Going four-dimensional, we add a time-stamp to all our claims. Instead of saying that it is snowing, say, we say that it is snowing at 4:37pm, Eastern Time, on December 31, 2011. Then, a statement about the future is true if it ends up true at the time. We need not see the logic as committing us to a determined future. We just know that statements about future events will eventually have truth values. There are also tense logics, which introduce temporal operators but maintain classical semantics, to help with time.

For failures of presupposition, M3, we can use Bertrand Russell's analysis of definite descriptions. In §3.11, there is a more-precise analysis of Russell's solution. For now, consider again an example of failure of presupposition.

WM The woman on the moon is six feet tall.

We can analyze WM to make the assumption explicit. We can re-cast WM as WM'

WM' There is a woman on the moon and she is six feet tall.

WM' has the form 'P • Q'. 'P' is false, so 'P • Q' is false. We can similarly recast WMN as WMN'.

WMN The woman on the moon is not six feet tall.
WMN' There is a woman on the moon and she is not six feet tall.

We regiment WMN' as 'P • ~Q'. P is false, so 'P • ~Q' is false. We thus do not have a situation in which the same proposition seems true and false. In both cases, P is false, so the account of the falsity of both sentences WM and WMN can be the same. We thus lose the motivation for introducing a third truth value.

M5 provides a reason for exploring the technical work of three-valued logics. But it gave us no philosophical reason for adopting them. For M4, nonsense, and M6, paradoxes, and M8, vagueness, we can deny that such sentences express propositions. We may claim that just as some strings of letters do not form words, and some strings of words do not form sentences, some grammatical sentences do not express propositions. This would be the same as to call them meaningless. This solution is a bit awkward, since it does seem that 'This sentence is false' is perfectly meaningful. But if it prevents us from having to adopt three-valued logics, it might be a useful move.

Exercises

1. Construct truth tables for each of the following propositions, under classical semantics and each of the three three-valued semantics (Bochvar, Kleene, Lukasiewicz). Compare the results.

1. P ∨ ~P
2. P ⊃ P
3. (P ⊃ Q) ≡ (~P ∨ Q)
   Note: you can construct the truth table for the biconditional by remembering that 'P≡Q' is logically equivalent to '(P ⊃ Q) • (Q ⊃ P)'

2. Use the indirect method of truth tables to test each of the following arguments for validity, under classical semantics and each of the three three-valued semantics (B, K_3, and L_3). Compare the results.

1. $P \supset Q$
   $P \quad /\therefore Q$

2. $P \quad /\therefore \sim(Q \bullet \sim Q)$

3. $P \quad /\therefore P \vee Q$

Paper Topics

1. Compare Bochvar semantics, Kleene semantics, and Lucasiewicz semantics. What differences do the different semantics have for classical tautologies? What differences do they have for classical inferences (validity and invalidity)? Be sure to consider the semantics of the conditional. Which system seems most elegant? This paper will be mainly technical, explaining the different semantics and their results.

2. Do three-valued logics solve their motivating problems? Philosophers explore three-valued logics as a way of dealing with various problems, which I discuss in these notes. Consider some of the problems and show how one of the systems tries to resolve the problem. For this paper, I recommend, but do not insist, that you focus on Kleene's semantics. If you try to deal with Epimenides, and the semantic paradoxes, you might want to focus just on that problem.

3. Bochvar introduced a new so-called assertion operator, $\vdash$. Use of this operator allows us to recapture analogs of classical tautologies within Bochvar semantics. Describe the truth table for this operator. Show how it allows us to construct tautologies. How does the new operator affect the set of valid formulas? (It can be shown that on Bochvar semantics, any argument using only the standard operators which has consistent premises, and which contains a sentence letter in the conclusion that does not appear in any of the premises, is invalid. You might consider this result, and the effect of the new operator on it.)

4. Quine, in Chapter 6 of *Philosophy of Logic*, calls three-valued logic deviant, and insists that to adopt three-valued logic is to change the subject. Why does Quine prefer classical logic? Consider his maxim of minimum mutilation. Who can deal better with the problems, sketched at the beginning of these notes, that motivate three-valued logic. (You need not consider all of the problems, but you should provide a general sense of how each approach works.)

5. Do assertions about the future have a truth value? Consider both the bivalent and the three-valued alternatives. You might compare Aristotle's view with that of Leibniz, who says that contingent truths are not necessary, even though they are certain. Alternatively, you could look at Haack's discussion of the way Aristotle's suggestion was pursued by Lukasiewicz. If you want to pursue an interesting technical discussion, Prior's "Three-Valued Logic and Future Contingents" is written in Polish notation.

6. How should we understand the sentence 'the king of America is not bald'? Consider Russell's theory of descriptions, and contrast it with Strawson's response. You might also consider the questions whether there a difference between logical and grammatical form, and, whether ordinary language has a logic.

7. Are there any people? Consider the problem of vagueness, and the many-valued approach to its solution.

Suggested Readings

Aristotle, *De Interpretatione.* In *The Complete Works of Aristotle, vol.1*, Jonathan Barnes, ed. Princeton University Press, 1984. On the sea battle, and future contingents.

Bochvar, D.A. "On a three-Valued Logical Calculus and Its Application to the Analysis of the Paradoxes of the Classical Extended Functional Calculus." *History and Philosophy of Logic* 2: 87-112, 1981.

Chomsky, Noam. *Syntactic Structures*. This book contains the discussion about colorless green ideas, but not a defense of three-valued logics. (Chomsky was arguing for a distinction between grammaticality and meaningfulness.)

Dummett, Michael. "The philosophical basis of intuitionist logic". In *Philosophy of Mathematics: Selected Readings*, 2nd ed., Paul Benacerraf and Hilary Putnam, eds. Cambridge University Press, 1983. And the selections by Heyting and Brouwer in the same volume. The intuitionists believed that an unproven mathematical statement lacked a truth value. These articles are all pretty technical, though.

Fisher, Jennifer. *On the Philosophy of Logic*. Chapters 7 and 9.

Haack, Susan. *Deviant Logic, Fuzzy Logic: Beyond the Formalism*. University of Chicago, 1996. Chapter 4 contains a discussion of Aristotle's view on future contingents, as well as more recent applications.

Haack, Susan. *Philosophy of Logics*. Cambridge, 1978. Chapters 9 and 11.

Kleene, Stephen. "On Notation for Ordinal Numbers." *The Journal of Symbolic Logic* 3.4: 150-5, 1938.

Leibniz, G.W. *Discourse on Metaphysics*. The early sections, especially §6-§13, contain his distinction between certainty and necessity.

Prior, A.N. "Three-Valued Logic and Future Contingents." *The Philosophical Quarterly* 3.13: 317-26, 1953.

Putnam, Hilary, "Three-valued logic" and "The logic of quantum mechanics", in *Mathematics, Matter and Method: Philosophical Papers, vol. 1*. Cambridge University Press, 1975. Do we need three-valued logic in order to account for oddities in quantum mechanics?

Quine, Willard van Orman. *Philosophy of Logic*, 2nd ed. Harvard University Press, 1986. The discussion of deviant logics and changing the subject is in Chapter 6, but Chapter 4, on logical truth, is exceptionally clear and fecund.

Quine, Willard van Orman. *The Ways of Paradox*. Harvard University Press, 1976. The title essay is the source of the 'yields a falsehood...' paradox, and contains an excellent discussion of paradoxes.

Read, Stephen. *Thinking about Logic*. Oxford, 1995. Chapters 6 and 7.

Russell, Bertrand. "On Denoting." In *The Philosophy of Language*, 5th ed., A.P. Martinich, ed. Oxford University Press, 2008. "On Denoting" is widely available.

Russell, Bertrand. *Introduction to Mathematical Philosophy*. Routledge, 1993. Chapter 16: "Descriptions". Contains a clearer discussion of Russell's solution to the problem of some forms of failure of presupposition than the ubiquitous "On Denoting."

Strawson, P.F. "On Referring." An alternative to Russell's theory of descriptions. Also in the Martinich collection.

Unger, Peter. "Why There Are No People." *Midwest Studies in Philosophy*, 1979.

Williamson, Timothy. *Vagueness*. Routledge, 1994. Chapter 1 has a nice discussion of the history of vagueness, and Chapter 4 discusses the three-valued logical approach to the problem.

See Priest's suggestions

§7: Truth and Liars

Philosophy is sometimes characterized as the pursuit of truth in its most abstract form. Given that characterization, it may come as no surprise that philosophers spend a fair amount of time thinking about the nature of truth itself. Unfortunately, philosophical discussions of truth can quickly become difficult and obscure. There is a lot of technical work on truth centering around responses to semantic paradoxes. There are also more fundamental questions about the nature of truth and our ability to know what is true. Technical work on the logic of truth is sophisticated, requiring subtle distinctions between object languages and metalanguages, sometimes many different metalanguages. Less-technical discussions of truth often rely on interpretations of the technical results.

We will start with a general overview of three non-technical theories of truth, asking whether truth is a property, and, if so, what kind of property. Then, we will look at the semantic paradoxes, and ask why they are important. Lastly, we examine Tarski's important work on truth, and his solution to the problems raised by the paradoxes.

4.7.1. Truth

The standard concept of truth is called the correspondence theory. The correspondence theory of truth traces back at least to Plato, though it is traditional to ascribe it to Aristotle.

> To say of what is that it is not, or of what is not that it is, is false, while to say of what is that it is, and of what is not that it is not, is true (*Metaphysics*, 1011b25).

According to the correspondence theory of truth, truth is a relation between words and the world. The truth of a sentence consists in its agreement with, or correspondence to, reality.

One worry about the correspondence theory is that we do not seem to have any extra-linguistic way to apprehend reality. If I want to compare, say, an elephant to a picture of an elephant, or a picture of a sculpture of an elephant to a picture of an elephant, I can hold both of them in front of me, gazing

from the one to the other.

If I want to compare my words to the world, I have to apprehend, on the one side, what the words mean, and on the other, the world. But, it has seemed to some philosophers, I only apprehend the world mediately, through my ideas of it. I do not have any access to the world as it is in itself.

Those of you who have worked through the epistemology of the modern era, especially the work of Locke and Berkeley and Hume, should understand the problem here. The correspondence theory says that truth is a matching between words and the world. But it seems as if I am unable to compare my words or my ideas to an independent world to decide whether there is a correspondence between them. I can only know about one side of the equation. I might be able to see whether my words match my ideas, but I am cut off from the world.

In response to such problems with the correspondence theory of truth, some philosophers have adopted coherence theories. According to coherentism, the truth of a sentence is a relation to other beliefs we hold. That relation is ordinarily taken to be consistency: a sentence is true if it is consistent with other sentences we hold. Different people apprehend the world in different ways, depending on their experiences, expectations, physiology, and background beliefs. The coherentist despairs of any method of resolving these inconsistencies among people and their beliefs.

For example, suppose that I believe in a traditional, monotheistic God and that you do not. GO will be true for me, since it coheres with my other beliefs.

GO God is omniscient.

In contrast, GO will be false for you, since it conflicts with your other beliefs. Since different people hold different beliefs, the coherence-truth of a sentence depends on the person who is considering the sentence. Coherence theories thus lead to relativism about truth. My truth will differ from your truth if my belief set is incompatible with yours.

The correspondence and coherence theories of truth each provide a univocal analysis of 'truth'. Insofar as they entail that there is a property called truth, they are both what are sometimes called inflationary theories of truth. Inflationary theories are distinguished from deflationary theories of truth. Deflationary theories of truth, developed in the last century, are often called minimalist theories. Deflationism has many proponents, and there are different ways of understanding and explaining the view. But deflationists are united in the belief that there is no essence to truth, no single reduction of truth to a specific property like correspondence or consistency.

Some deflationists claim that truth is just a device for simplifying long conjunctions. If you said a lot of smart things at the party, I could list them all. Or, I could just assert LN.

LN Everything you said last night was true.

In LN, 'true' is eliminable by a long set of sentences listing all of what you said last night. Such eliminations are, according to the deflationist, the central purpose of 'truth'. Otherwise, 'truth' is merely a redundant term. Indeed, deflationism is often called a redundancy theory of truth: to say that 'snow is white' is true is just to say, redundantly, that snow is white.

Both inflationists and deflationists agree that a minimal condition for truth is what we call the T-schema, or Convention T, following Tarski.

CT p is true if and only if x

In CT, 'p' is the name of any sentence, and x are the truth conditions of that sentence. We can use CT to specify the truth conditions for any sentence. Here are some instances of the T-schema.

CT1	'The cat is on the mat' is true if and only if the cat is on the mat.
CT2	'2+2=4' is true if and only if 2+2=4
CT3	'Barack Obama is president' is true if and only if the husband of Michelle Obama and father of Sasha Obama and Malia Obama is head of the executive branch of the United States of America.

Note that, as in CT3, the truth conditions on the right of the 'if and only if' need not be expressed in the same terms as the sentence on the left. We can even use a different language for the sentence and for its truth conditions, as in CT4.

CT4 'El gato está en el alfombrilla' is true if and only if the cat is on the mat.

You could, in principle, understand the truth conditions of CT4 without understanding the meaning of the Spanish sentence on the left side.

Inflationists and deflationists disagree about whether CT is all there is to know about truth. The inflationist believes that there are explanations of the concept of truth inherent in the truth conditions on the right side of CT. For the correspondence theorist, 'the cat is on the mat' is true because there is a cat, which corresponds to 'the cat', and there is a mat, which corresponds to 'the mat', and there is a relation, being on, which the cat and the mat satisfy, or in which they stand. All other instances of the T-schema will have similar explanations in terms of the correspondence of words to worlds.

The deflationist, in contrast, believes that the T-schema is all there is to know about truth, and that there is no single kind of explanation of why all sentences are true. 'Truth' varies in application. The explanation of the truth of CT1, for example, must differ significantly from the explanation of the truth of CT2. Our justification for asserting statements about cats and mats relies, often, on direct observation. Our justification for asserting statements about mathematical objects relies only indirectly (at most) on sense experience. To repeat, according to the deflationist, we do not even need 'true' in our language. It's just a handy tool.

Deflationists look at CT as a satisfactory definition of truth. That's why deflationism also goes by the name 'redundancy theory'. Inflationists about truth look at CT as merely a minimal condition for truth. They claim that there are additional requirements, like correspondence to reality.

Alfred Tarski, one of the great twentieth-century logicians, introduced CT as an essential component of his treatment of the semantic paradoxes. There is some debate about whether Tarski is best understood as a deflationist or as an inflationist, to which we will return after looking in more detail at how he dealt with the paradoxes.

4.7.2. The Liar, and Other Semantic Paradoxes

Much work on truth over the last century has been technical, in response to the semantic paradoxes. The most important semantic paradox is called the liar.

L This sentence is false.

L is an example of a paradoxical sentence. If L is true, then, since it says that it is false, it must be false. But if it is false, then since it says that it is false, it must be true. But, if it is true... L thus lacks a single, definite truth value, even though it is a grammatically well-formed sentence.

The liar is often called Epimenides' paradox. Epimenides was a Cretan to whom the statement that all Cretans are liars is attributed.

W.V. Quine, in his essay "The Ways of Paradox," argues that there are grounds to question either the paradoxicality or the well-formedness of L. It is not clear what 'this sentence' refers to. If we substitute 'this sentence is false' for 'this sentence', then we get LQ.

LQ 'This sentence is false' is false.

LQ does not ascribe falsity to itself, and the paradox is avoided, or at least delayed. Still, we can find other, similarly troublesome sentences. Quine constructed QP, which avoids the above problem.

QP ‘Yields falsehood when appended to its own quotation’ yields falsehood when appended to its own quotation.

In both L and QP, the culprit seems to be the invocation of the concept falsity. Truth and falsity are called semantic terms. ‘Semantic’ can refer to truth or to meaning. In ordinary usage, when we talk about semantics, we refer to meanings. When we present a semantics for a formal language, we provide truth conditions for the wffs of the language. Our semantics for **PL**, for example, consisted of truth tables. The more-complicated semantics for **M** consist of interpretations and satisfaction conditions, and ultimately on truth.

The problem with many sentences like L and QP seems to be rooted in the presence of semantic terms, like ‘true’ and ‘false’. Thus, such problematic sentences are called semantic paradoxes.

One diagnosis of many semantic paradoxes, including the liar, is that they involve illicit self-reference. Another self-referential paradox, the barber paradox, is due to Bertrand Russell, though he credits an anonymous source. Consider the barber in a town who shaves all the men who do not shave themselves. Does he shave himself? You can construct a puzzling declarative sentence, similar to the liar, which I leave to you as an exercise.

Not all semantic paradoxes involve truth, or self-reference. Consider Grelling’s paradox. Some predicates apply to themselves, whereas others do not. ‘Polysyllabic’ is polysyllabic; ‘monosyllabic’ is not monosyllabic. Call a predicate heterological if it does not apply to itself. ‘Monosyllabic’ is heterological; ‘polysyllabic’ is not heterological. (We can call it autological, or homological.) Now, consider whether ‘heterological’ applies to itself. If it does, then ‘heterological’ is not heterological. But, if ‘heterological’ is not heterological, then it does not apply to itself, which means that it is heterological. We can construct a statement involving ‘heterological’ whose truth value is puzzling.

HH ‘Heterological’ is heterological.

Grelling’s paradox is semantic, but does not involve ‘truth’ or ‘falsity’ explicitly. Grelling’s paradox is about meaning.

S1 and S2 are two popular solutions to the semantic paradoxes.

S1 Introduce a third truth value for paradoxical sentences.
S2 Banish semantic terms from formal languages.

There are two problems with S1. First, systems of three-valued logic either lose logical truths and valid inferences or ascribe truth to conditional sentences with indeterminate antecedents and consequences. Second, adding a third truth value will not solve the problem of the strengthened liar.

SL This sentence is not true.

If SL is true, then since it says that it is not true, it must be either false or indeterminate. But, if it is false or indeterminate, then what SL says holds of itself. So, SL is true. The paradox recurs.

The second popular solution, S2, is Tarski’s, which we will examine in a moment. First, we should look in greater detail how the paradoxes create logical difficulties.

4.7.3. Explosion, or What's So Bad About the Paradoxes?

In the early twentieth century, truth had gotten a terrible reputation, in large part due to the paradoxes. The paradoxes lead to contradictions. Contradictions are unacceptable in traditional, or classical, formal systems because a contradiction entails anything. This property of classical systems is called explosion.

Explosion	1. P • ~P	/ Q
	2. P	1, Simp
	3. P ∨ Q	2, Add
	4. ~P • P	1, Com
	5. ~P	4, Simp
	6. Q	3, 5, DS

QED

To see how the liar leads to a contradiction, consider L again. Applying the T-schema yields CTL.

CTL L is true if and only if L is false.

We can translate this sentence into **M** by taking a constant, say 'p', to stand for the sentence L, and introducing a truth predicate, 'Tx'. We also have to take 'P is true' to be the negation of 'P is false'; the strengthened liar will work a bit differently.

1. Tp ≡ ~Tp	From CT and the definition of 'p'
2. (Tp ⊃ ~Tp) • (~Tp ⊃ Tp)	1, Equiv
3. (~Tp ⊃ Tp) • (Tp ⊃ ~Tp)	2, Com
4. ~Tp ⊃ Tp	3, Simp
5. ~~Tp ∨ Tp	4, Impl
6. Tp ∨ Tp	5, DN
7. Tp	6, Taut
8. Tp ⊃ ~Tp	2, Simp
9. ~Tp ∨ ~Tp	8, Impl
10. ~Tp	9, Taut
11. Tp • ~Tp	7, 10, Conj

Tilt!

Our natural language contains the word 'true', as a predicate. If we include a truth predicate in our formal language, we can construct the liar sentence. If we can construct the liar sentence, we can formulate an explicit contradiction. Contradictions explode. Everything is derivable. But, we know that not every sentence is true. So, if we include a truth predicate in our formal language, our formal language will not be able to contain, or reveal, our true commitments.

The excitement surrounding the new logic of the early twentieth century included hopes that all human knowledge could be represented by formal languages, like the logic we are studying. Since contradictions lead to explosion, and formal languages in which the paradoxes are representable lead to contradictions, it became seen as essential to avoid formalizing the concept of truth. Since formal languages were seen as the locus of all of our knowledge, it seemed that truth was just not a legitimate term, not something that we could know.

The bad reputation of truth explains, at least in part, the interest of many philosophers in the relativism of coherence truth. All recent work on truth, whether deflationary or inflationary, owes its origins to Tarski, who, in the 1930s, showed how to rehabilitate the concept of truth within formalized languages, how to avoid explosion without giving up on a formalized notion of truth.

4.7.4. Tarski's Solution

Tarski's solution to the liar paradox is to distinguish between an object language and a metalanguage, and to rule sentence L out of the object language. The object language is the language in which we are working. The metalanguage is a language in which we can talk about the object language. Instances of the T-schema are sentences of the metalanguage which we can use to characterize truth for the object language. Since we are constructing a formal object language, like **PL** or **M**, we can include or exclude any terms we wish. One of the advantages of constructing a formal language is that we can make it as clean and clear as we wish. We can omit 'true' from our object language and thus avoid the lair paradox.

Deleting just the liar from the object language might appear arbitrary and *ad hoc*. Tarski claims that the paradoxes show that all uses of the term 'true', and related semantic terms, are illegitimate within any object language. We can not construct the truth predicate for a language within that language because that would lead to a contradiction. Within a metalanguage, we can construct a truth predicate for any object language. But that truth predicate will be an element only of the metalanguage, not of the object language itself. We have no term in **M** for truth in the way that we have a term for identity.

Tarski's solution to the semantic paradoxes forces us to step out of our object language to examine its sentences for truth or falsity. To determine which sentences of an object language are true and which are false, we examine the truth conditions as given on the right hand side of instances of the T-schema. While the sentences themselves are elements of the object language, the truth conditions are, technically, written in the metalanguage.

The key to Tarski's solution to the liar paradox is that sentences like L are ill-formed because they include 'false' in the object language. When I want to use a sentence like LN, he claims, I implicitly ascend to a metalanguage to do so. In a metalanguage, I can also construct sentences like the important TC.

TC All consequences of true sentences are true.

Sentences like TC are fundamental to metalogic, and model theory, fields that Tarski more or less created. In metalogic, we explore questions of whether a formal system is sound, or complete, or decidable. We will put them aside, here, and see if Tarski's T-schema CT can help us understand our ordinary conception of truth.

Before we approach the questions about whether Tarski's theory is inflationary or deflationary, about how we are to understand the concept of 'truth' more broadly, we should look briefly at a technical concern about the sufficiency of his solution, and an alternative.

4.7.5. Kripke's Alternative

Tarski's construction produces a hierarchy of languages. To construct a truth predicate for an object language, we eliminate semantic terms from our object language and ascend to a metalanguage. For our purposes, that means that we will not include a truth predicate as part of **M**. If we want to know about the truth of sentences of **M**, we take an external perspective, working in a metalanguage.

We might reasonably wonder about truth in the metalanguage. Of course, for the same reasons that the object language can not contain a truth predicate, the metalanguage can not contain its own truth predicate. But, we can construct a truth predicate for the metalanguage in a further metalanguage. To construct a truth predicate for the second metalanguage, we can construct a third, and so on. Each of the separate truth predicates occurs at a different level in this ever-expanding hierarchy.

The relations among these truth predicates are merely analogical. Each metalanguage is distinct, and has different terms. Each truth predicate is independent of each other. We are burdened not with one 'truth', but an infinite hierarchy of 'truth's.

To make matters worse, there are cases in which we do not know which level in the hierarchy any particular use of 'true' or 'false' belongs to. Consider GB.

GB Everything George W. Bush says is false.

GB must be made in a metalanguage one step higher than anything that Bush ever said. If I assert GB, then to know what level my 'false' belongs to, I need to know about all the levels of Bush's uses of 'true' and 'false'. If Bush once claimed BC, for example, then in order to know what level Bush's 'false' occurs at, we also need to know all the levels of the uses of 'true' and 'false' in whatever Clinton said.

BC Everything Bill Clinton says is false.

Furthermore, if Clinton were the speaker of GB, then Bush and Clinton become embroiled in a semantic circle. The level of GB must be higher than the level of BC. The level of BC must be higher than that of GB. Tarski's hierarchical approach seems to lead to a contradiction in this case despite the fact that there seems to be nothing contradictory about the conjunction of GB (stated by Clinton) and BC (stated by Bush). They are just both false statements: both Clinton and Bush have uttered some truths.

Saul Kripke, in a paper called "Outline of a Theory of Truth," showed that we can construct a truth predicate for a language embedded within the object language itself, without creating paradox. Here is a quick sketch of Kripke's approach.

We start with a base language, containing no logical connectives, quantifiers, or truth predicate. Then, we add a truth predicate to the language itself. We can more-or-less easily decide which sentences of the base language are true and which are false, since there is no truth predicate in the base language. Then, we can add the familiar logical connectives: negation, conjunction, and disjunction, say. The semantics for the propositional connectives are easily presented, as well. So, we can apply the truth predicate to all base-level sentences and logical functions of them.

Next, we can consider sentences with single uses of semantic terms licensed so far. We repeat the original process, adding more complex sentences to our lists of true and false sentences. We can proceed to sentences of greater and greater semantic complexity. At each level, we bring along all the earlier sentences, and apply the truth predicate to them. But, the truth predicate does not apply to sentences at its own level. Eventually, we can, in principle, reach any of a variety of fixed points past which further construction is unwarranted. There are many different fixed points, and lots of technical work can be done with them.

The key to Kripke's construction is that he produces an object-level truth predicate. This object-language truth predicate allows us to value many sentences that include 'true'. We can have a language with all of those sentences, and one truth predicate for all of them. Kripke contains the entire Tarskian hierarchy in one language.

Kripke's paper is technical, but extremely elegant. Those with mathematics or other technical backgrounds might enjoy working on a paper about it. For now, I will put aside Kripke's improvements on Tarski's technical work, and the problems of the hierarchy of truth predicates, and return to the philosophical questions about truth raised by Tarski. Most importantly, how does the technical work on truth illuminate the deeper issues about the nature of truth?

4.7.6. Is Truth Deflationary or Inflationary?

Tarski calls the notion of truth which underlies his method of introducing a truth predicate into a metalanguage the semantic conception of truth. He uses sentences like LN and TC to show that ‘truth’ plays an essential role in a theory. It might thus seem like Tarski is an inflationist, indeed a correspondence theorist.

But Tarski’s claim that ‘truth’ is essential may not have inflationary implications. If ‘true’ is a device used to refer to other sentences, it depends on what we think of those other sentences, the ones without ‘true’ and with content. If we need a words-worlds relation in order to ascribe ‘true’ to a sentence, then truth will not be merely deflationary, or redundant. If all there is to truth is eliminable, then perhaps there is no essence to truth. Even Aristotle’s original claim could itself be given a deflationary interpretation! Tarski prescribes a method to determining the correct notion of truth.

> It seems to me obvious that the only rational approach to [questions about the correct notion of truth] would be the following: We should reconcile ourselves with the fact that we are confronted, not with one concept, but with several different concepts which are denoted by one word; we should try to make these concepts as clear as possible (by means of definition, or of an axiomatic procedure, or in some other way); to avoid further confusions, we should agree to use different terms for different concepts; and then we may proceed to a quiet and systematic study of all concepts involved, which will exhibit their main properties and mutual relations (355).

Furthermore, Tarski believes that the semantic conception is agnostic among any deeper philosophical debates.

> We may accept the semantic conception of truth without giving up any epistemological attitude we may have had; we may remain naive realists, critical realists or idealists, empiricists or metaphysicians - whatever we were before. The semantic conception is completely neutral toward all these issues (362).

Hartry Field’s paper “Tarski’s Theory of Truth” argues convincingly that Tarski is not a deflationist. It is, like Kripke’s paper, difficult and technical, but influential and fecund. It is more strictly philosophical than Kripke’s paper. Field shows that in order to use the T-schema as a definition of truth, we need to supplement it with some kind of account of why we choose certain sentences to be true and not others. To see the problem, remember that we could understand the truth conditions in CT4 without understanding the Spanish sentence on the left. To capture truth, it is not enough just to list the true and false sentences of a language. We want to analyze the component parts of the Spanish expressions, and how they interact to form true or false sentences. CT, by itself, does not provide that kind of explanation. Tarski’s construction only reduces ‘truth’ to other semantic notions.

If we are merely concerned with constructing a metalinguistic truth predicate, CT might suffice. We might, in contrast, wish to take Tarski’s claim to a semantic notion of truth seriously. In that case, we need not merely to explain truth in terms of other semantic notions, but to show how sentences become either true or false. We would like, in addition to CT, an explanation of why the terms are true of the things of which they are true, in a way that is consistent with our other scientific commitments. It is not that we could not add such an account to complete Tarski’s theory. But, once we do, the theory does not appear deflationary.

We started by wondering about the nature of truth, whether it is correspondence to reality, or consistency, or whether it lacks any univocal nature. The question has now become whether Tarski’s formalized semantic conception captures our ordinary notion. Is there more to be said about truth than Convention T?

4.7.7. Did Tarski Present the Final Word on Truth?

There are at least two ways to look at Tarski's semantic theory of truth. The first way is minimalist, and it focuses on the condition of adequacy, the schema CT. The second way is inflationist, and it focuses on the extent to which Tarski legitimizes our ordinary, correspondence notion of truth. There is no question that the notion of truth is useful, in sentences like LN, and essential to metalogical work, in sentences like TC. Tarski, and those following him, have vindicated formal theories of truth insofar as they allow us to capture these minimal uses of the term.

The question of whether philosophers need an inflationary notion of truth continues to be debated. Many philosophers, Tarski among them, believe that science aims at truth. The main reason we want consistent theories is because we know that an inconsistent theory contains a falsehood.

There are obvious epistemic worries about our access to truth, our ability to know what is true and what is false. The old problem of whether we can assess a words-worlds connection still resonates. Some philosophers continue to try to replace truth with a weaker condition like warranted assertability or coherence.

As I mentioned, much of the contemporary work on truth and the paradoxes is technical, though the classical discussion of theories of truth is mainly philosophical. Michael Lynch's *True to Life: Why Truth Matters* is a friendly introduction to the non-technical work on truth by someone who has worked with the contemporary questions.

One of the more controversial but productive areas of recent research has been dialetheism. According to dialetheists like Graham Priest, the liar is both true and false. There has been a lot of technical work on paraconsistent logics, logics which contain contradictions. Contradictions in classical logic are explosive: anything follows. So, dialetheists look to block explosion in a variety of ways.

Whether or not Tarski's solution to the problem of the paradoxes is ideal, the distinction between object language and metalanguage has become fundamental in all contemporary treatments of logic. In this textbook, I have carefully presented precise rules for the formation of our object language, which is our proper domain of study. The Greek letters I use to describe argument forms, and the truth values 1 and 0, are all elements of the metalanguage we use to study the object language. The formalization of this distinction traces directly to Tarski's work on truth.

Paper Topics

1. Does introducing a third truth-value solve the problem of the liar? Discuss the strengthened liar paradox. Kirkham has a good, if brief, discussion of the strengthened liar.

2. Is truth deflationary or inflationary? See Horwich, and the Lynch collection. Fisher has a fine introductory discussion.

3. Is truth relative to a language? Tarski's definition of truth introduces a different truth predicate for each language, and creates a hierarchy of languages. Is this construction objectionable? See Fisher, Kirkham, and the Lynch collection.

4. Graham Priest has lately been defending dialetheism, the claim that there can be true contradictions. Can there be true contradictions? Is the liar one of them?

5. For a more technical paper, describe the difference between Kripke's truth predicate and Tarski's hierarchy. What advantages does Kripke claim for his construction? Is it satisfactory?

6. Is truth a correspondence between words and reality? See the Lynch collection for the classic, historical discussion.

Suggested Readings

Aristotle. *Metaphysics.* In *The Complete Works of Aristotle, vol.2*, Jonathan Barnes, ed. Princeton University Press, 1984. The definition of truth is at 1011b25.

Beall, J.C. *Liars and Heaps: New Essays on Paradox*. Oxford: Clarendon Press, 2003.

Field, Hartry. "Tarski's Theory of Truth." In his *Truth and the Absence of Fact*, Oxford, 2001.

Field, Hartry. *Saving Truth from Paradox*. Oxford, 2008. Field is astoundingly good, but only accessible to the most ambitious readers.

Fisher, Jennifer. *On the Philosophy of Logic*, Wadsworth, 2008. See Chapter 3.

Haack, Susan. *Philosophy of Logics*. Chapters 7 and 8.

Horwich, Paul. *Truth*. Oxford University Press, 1999.

Kirkham, Richard. *Theories of Truth: A Critical Introduction*. The MIT Press, 1995.

Kripke, Saul. "Outline of a Theory of Truth." In Jacquette, *Philosophy of Logic*.

Künne, Wolfgang. *Conceptions of Truth*. Oxford: The Clarendon Press, 2003.

Lynch, Michael. *The Nature of Truth: Classic and Contemporary Readings*. The MIT Press, 2005. A great collection of the most important views.

Lynch, Michael. *True to Life: Why Truth Matters*. The MIT Press, 2005. A fairly simple introduction.

Priest, Graham. *In Contradiction.* Dordrecht: M. Nijhoff, 1987.

Priest, Graham. *Beyond the Limits of Thought*. Oxford: Oxford University Press, 2002.

Priest, Graham, J.C. Beall and Bradley Armour-Garb. *The Law of Non-Contradiction: New Philosophical Essays*. Oxford: Clarendon Press, 2004. See especially the Introduction, and Essays 1, 7, and 12.

Quine, Willard van Orman. "The Ways of Paradox." Harvard University Press, 1976. The title essay is the source of the 'yields a falsehood...' paradox, and contains an excellent discussion of paradoxes.

Tarski, Alfred. "The Semantic Conception of Truth and the Foundations of Semantics." *Philosophy and Phenomenological Research* 4.3: March 1944, pp 341-376.

§8: Quantification and Ontological Commitment

4.8.1. Grammar, Logic, and the Ontological Argument

All of our first-order languages contain two types of quantifiers, existential and universal. The existential quantifier is sometimes called the particular quantifier. Calling '∃' an existential quantifier seems to assume that uses of the quantifier in our logic involve us in commitments to the existence of something or other. Calling '∃' a particular quantifier leaves open the option that our logic may be interpreted without such commitments. We sometimes call the collection of objects which we believe to exist our ontological commitments. In this section, I will discuss ontological commitment, the way that language hooks on to the world, and the connection between these topics and the quantifiers.

Q1 and Q2 are basic questions which have occupied philosophers for a long time.

Q1	What exists?
Q2	How do we know?

Q1 starts us on the road to metaphysics: Are there minds? Are there laws of nature? Is there a God? The objects on our list of what we think exists are called our ontology, or our ontological commitments. Some kinds of things more-or-less obviously exist: trees and houses and people. The existence of other kinds of things is more contentious: numbers, souls and quarks. Even the question of the existence of some particular things can be debated: does Descartes, for example, exist?

Q2 starts us on the road to epistemology. If we believe a claim, like the claim that there are minds in the world in addition to bodies, then we should have some reasons for believing that claim. Answers to Q1 are thus tied to answers to Q2. If I claim that electrons exist, I should be able to demonstrate how I discovered them, or how I posited them, or how their existence was revealed to me. If you deny my claim that the tooth fairy exists, you will appeal the fact that we never see such a thing, for example. To resolve disputes about what exists, we should have a method to determine what exists. At least, we should agree on a way to debate what exists. Our answer to the question whether Descartes exists, for example, depends on what we say about all people who are no longer alive.

Since Descartes is no longer alive, we could naturally think of him as not existing. But, we could think of the world as four-dimensional, with three spatial-dimensions and one temporal dimension. What exists, then, is everything that has any spatial and/or temporal coordinates. In a four-dimensional world, people are extended (temporally) through some portion of the world. It is typical to call such a conception a space-time worm. Whether a person's space-time worm is present at any particular temporal point, like this one right now, is irrelevant to her/his existence.

We are faced with a question of whether to conceive of the world three-dimensionally, in which case Descartes does not exist, or four-dimensionally, in which case Descartes does exist. Metaphysics is the study of our answers to such questions. Epistemology is the systematic study of the reasons one has for choosing one or the other. It is the study of justification of our beliefs.

Frege, in addition to developing modern logic, contributed centrally to what has become known as the linguistic turn in philosophy. In the twentieth century, many philosophers turned to the study of language in order to engage metaphysical and epistemological questions. If we could become clearer about how language works, some philosophers believed, we could answer some of our long-standing metaphysical and epistemological questions.

One of the early insights made by philosophers of language, perhaps properly ascribed to Bertrand Russell, is that grammatical form is not a sure guide to logical form. Grammatically, for example, KE and KP are parallel.

KE	Saul Kripke exists.
KP	Saul Kripke is a philosopher.

Both contain a term for the same subject, Saul Kripke. The first contains a grammatical predicate of existence. The second contains a grammatical predicate of being a philosopher. But, in first-order logic, we regiment KP using a predicate while we regiment KE using an existential quantifier.

KEr	$(\exists x)x=s$	or	$(\exists x)Sx$
KPr	Ps	or	$(\forall x)(Sx \supset Px)$

Thus, our typical regimentations of natural languages into first-order logic presupposes that predications of existence are really different from other kinds of predications.

Philosophers have argued for a long time about grammatical predications of existence. Many of those debates have focused on an argument for the existence of God called the ontological argument. The ontological argument traces back at least as far as St. Anselm in the eleventh century, and was central to Descartes's *Meditations*, and the works of Spinoza and Leibniz and many other philosophers. There are various forms of the ontological argument. OA captures its core.

OA

OA1. The term 'God' may be used to stand for the concept of a thing with all perfections, without presuming that God exists.
OA2. Existence is a perfection; it is perfect to exist while not-existing would be an imperfection.
OA3. The claim that God does not exist, then, would be a contradiction.
OAC. God exists.

In addition to its many defenders, the ontological argument has had many detractors. OA begins with a simple definition and quickly concludes, from an analysis of the logic of our language, that God exists. That conclusion has seemed to many philosophers far too quick. Since David Hume's time, in the eighteenth century, a standard objection to the ontological argument has focused on the difference between existence and other kinds of predications.

> Though certain sensations may at one time be united, we quickly find they admit of a separation, and may be presented apart. And thus, though every impression and idea we remember be considered as existent, the idea of existence is not derived from any particular impression.
>
> The idea of existence, then, is the very same with the idea of what we conceive to be existent. To reflect on any thing simply, and to reflect on it as existent, are nothing different from each other. That idea, when conjoined with the idea of any object, makes no addition to it. Whatever we conceive, we conceive to be existent. Any idea we please to form is the idea of a being; and the idea of a being is any idea we please to form (Hume, *Treatise on Human Nature*, Book I, Part II, §VI).

Hume's allegation, here, is that our predications of existence are naturally to be taken as unique kinds of predications. They do not augment their subject. They just repeat the idea of the subject. According to Hume, saying that Saul Kripke exists is to say nothing more than 'Saul Kripke'. To say that God exists is not to ascribe any further property. We can not, merely by analyzing our language, conclude whether something exists or not.

Immanuel Kant, following Hume and echoing an objection made to Descartes by Pierre Gassendi, argued that existence is not a real predicate. In KE, we are predicating (grammatically) existence of the statue. But, we are not saying anything substantive about Kripke. In KP, we use a real predicate. Any property can be predicated of any object, grammatically. SM is a grammatical sentence, even if it is nonsensical.

SM Seventeen loves its mother.

'Loves one's mother' is a real predicate. But Kant's point is that one can not do metaphysics through grammar alone. Existence is a grammatical predicate. It works in natural language like a predicate. But it is not a real predicate. When we want to analyze precisely what we believe, we do not take claims of existence as ascribing real property. So grammatical form is not the same as logical form.

The claim that existence is not a predicate becomes manifest in contemporary first-order logic. Properties like being a god, or a person, or being mortal or vain, get translated as predicates. But existence is taken care of by quantifiers, rather than predicates. To say that God exists, we might use one or other version of GE.

GE $(\exists x)x=g$ or $(\exists x)Gx$

The concept of God, on the left, and the object, on the right, are represented independently of the claim of existence. If we take first-order logic to be our most austere, canonical language, we vindicate the claim, from Gassendi and Hume and Kant and Russell, that existence is not a real predicate, that grammatical form is distinct from logical form.

In first-order logic, then, it is typical to innoculate constants and predicates from questions about existence. Questions about existence are all focused on the quantifiers. This view leads directly to a worry about names without bearers. If Hume and Kant are correct about names, then the proper analyses of EB and EBN are puzzling.

EB The Easter Bunny exists.
EBN The Easter Bunny does not exist.

Terms such as 'the Easter Bunny' are called non-referring singular terms. The problem with non-referring singular terms is that if names merely denote objects, then in order for EB and EBN to be sensible, 'the Easter Bunny' seems to have to refer to something. Not only are EB and EBN sensible, but they have clear and uncontroversial truth values: EB is false and EBN is true.

The most obvious, indeed the only plausible, thing for EB and EBN to refer to is the Easter Bunny. But, there is no such thing. Hume claimed, "To reflect on any thing simply, and to reflect on it as existent, are nothing different from each other." But, when I reflect on a non-referring singular term, I am not reflecting on it as existent. Indeed, it seems part of our very concept of the Easter Bunny that it is non-existent.

Some kinds of uses of singular terms, like references to myself, may presuppose the existence of an object named. Other kinds of uses, like references to the Easter Bunny, do not. Hume argued that predications of existence of a subject presuppose the existence of the term to which the subject exists. EB and EBN seem to be counterexamples to Hume's claim.

We started this section by asking about the meaning of the existential quantifier and the relation between our logic and our ontology. One obvious place to look for answers to ontological questions is at the names used in a theory. But, names can be misleading. In the twentieth century, many philosophers turned their attention away from names and toward existential quantification as the locus of our expressions of ontological commitment.

4.8.2. The Riddle of Non-Being

W.V. Quine was among the most impassioned defenders of the connection between existential quantification and existence claims. In his important article defending that connection, “On What There Is,” Quine focuses on the non-referring name ‘Pegasus’.

NP There is no such thing as Pegasus.

Part of Quine’s worry is semantic. How can I state NP, or any equivalent, without committing myself to the existence of Pegasus? If we take existence to be a predicate, NP says that there is some thing, Pegasus, that lacks the property of existence.

NPr1 ~Ep

But Pegasus is not anything, ‘p’ does not refer, and I can not say something about nothing. So, if Pegasus does not exist, then it seems a bit puzzling how I can deny that it exists. I am talking about a particular thing. It has to have some sort of existence in order for NP to be sensible.

One option for understanding the term ‘Pegasus’ is to take it to refer to the idea of Pegasus. John Locke, and many of the modern philosophers who followed him, took words to stand for ideas in our minds. If ‘Pegasus’ refers to my idea of Pegasus, we can best understand NP as claiming that the idea is not instantiated.

But taking names to refer to the ideas we associate with those names demonstrates a basic confusion of ideas and objects. ‘Benedict Hall is a warm building’ refers to an object, not an idea. ‘Pegasus is a winged horse’ has the same grammatical structure. Why would it refer to an idea, rather than an object? How do we know when a name refers to an object, like Benedict Hall, and when it refers to an idea of an object, like ‘Pegasus’? Some singular terms appear to refer to objects even though we do not know if those objects really exist. We do not know whether there is life on other planets, but ‘the first planet discovered by people on Earth which supports life’ does seem like a sensible term.

More importantly, against the suggestion that terms like ‘Pegasus’ refer to my idea of Pegasus, I do have an idea of Pegasus. If NP referred to my idea, then it would, on a natural interpretation, be false. The problem with NP is that it is true just because there is no object in the world corresponding to my idea.

Another option for understanding terms like ‘Pegasus’ distinguishes between existence and subsistence. According to the doctrine of subsistence, all names of possible objects refer to subsistent objects. But only some names refer to existent objects. The German philosopher Alexis Meinong defended the distinction between existence and subsistence. Meinong would say that since Pegasus subsists, statements like NP can be truly interpreted as saying that Pegasus merely subsists and does not exist.

One problem for Meingong’s solution to the puzzle is that we also have terms for impossible objects, like a round square cupola. We might take terms for impossible objects to be meaningless. But if we take ‘round square’ to be meaningless, even though ‘round’ and ‘square’ are meaningful, we have to abandon the compositionality of meaning, that the meanings of longer strings of our language are built out of the meanings of their component parts. ‘Round’ is meaningful and ‘square’ is meaningful, but ‘round square’, since there can be no such thing, is meaningless.

Quine says that the abandonment of meaning for such terms is ill-motivated. But his main argument consists of his positive account of how to deal with names which lack referents, and how to deal with debates about existence claims, generally.

4.8.3. Quine's Method

One method for determining what we think exists, a method favored by Locke and Hume and Quine's mentor Rudolf Carnap, relies on sense experience. For these philosophers, all claims about what exists must be justified directly by some kind of sense experience. The claim that my knowledge of oranges must derive from my experiences of oranges seems plausible enough. Further, we could use the same claims to defend our beliefs that there is no Loch Ness monster: no one has any direct sense experience of Nessie. But these empiricists had difficulty explaining our knowledge of mathematics and atoms. We do not have any sense experience of the abstract objects of mathematics, and yet we know many facts about them. We have only the merest and most indirect sense experience of atoms.

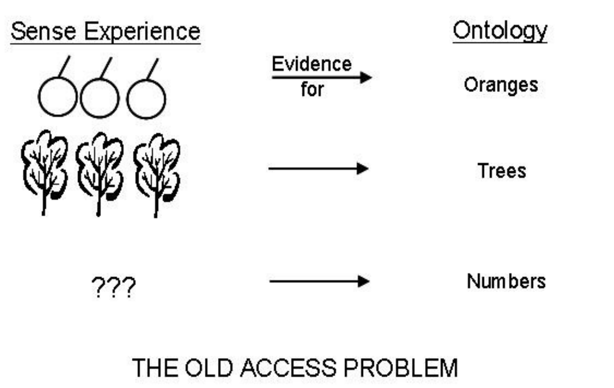
Another method for determining our beliefs about what exists is favored by Descartes and the great logician Kurt Gödel. These rationalists rely on human reasoning in addition to sense experience. Rationalists have an account of our beliefs about numbers, since they are object of our pure thought. But rationalists are often accused of mysticism. Indeed many rationalists, historically, claimed to have certain knowledge of the existence of God. A seemingly magical ability to know something independently of sense experience can be used to try to justify beliefs in ghosts and spirits, as well as numbers and electrons.

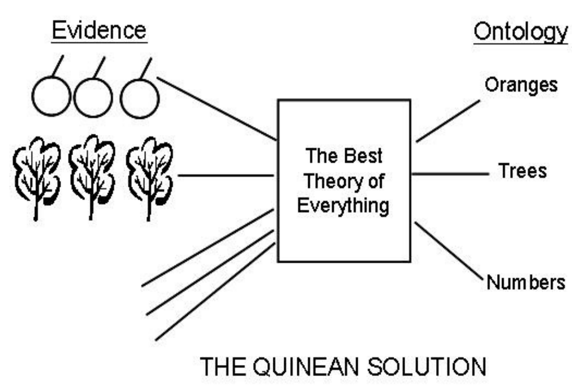
Quine's method for determining ontic commitments uses the tools of first-order logic.

To be is to be the value of a variable (Quine, "On What There Is" 15).

I will attempt to answer two questions about Quine's method. First, what variables are relevant to the question of what exists? Second, what does it mean to be a value of a variable?

The answer to the first question is fairly straightforward. Quine is concerned with the best theories for explaining our sense experience. Quine is thus much like his empiricist predecessors in narrowing his focus on sense experience. But, he is unlike traditional empiricists in that he does not reduce all claims of existence directly to sense experiences. Instead, Quine constructs a theory of our sense experience. Then, he looks at the theory, and decides what it presupposes, or what it posits. Our best ontology will be derived from our best theory.





We might encounter many difference, competing theories of the world. Determining which theory is best is tricky. We want our theories to be simple and elegant, and yet explanatorily powerful. We want to unify various diverse phenomena. Even among the most powerful and elegant best theories, there may be competitors. Quine, indeed, raised questions about whether our best theories are physicalistic or phenomenalist. A physical theory makes claims about the external, material world. A phenomenalist theory makes claims about our experiences of the world. Since we know directly only our own experiences, perhaps our best theory should refer only to our experiences, not to some posited causes of those experiences. Should we commit only to the experiences we have, or to the physical world which we ordinarily think causes our experience? In any case, the best theory will have to have some relation to the best science we can muster.

Even given an ability to choose among competing theories, there are questions about how to formulate and read a theory. Quine urges that the least controversial and most effective way of formulating a theory is to put it in the language of first-order logic. He motivates his appeal to first-order logic with a discussion of Russell's theory of definite descriptions.

Consider, 'The King of America is bald'. If we regiment 'the king of America' as a name, using a constant, then we are led to the contradiction KO.

KO $\quad\quad$ $\sim Bk \bullet \sim\sim Bk$

We assert '$\sim Bk$' because the sentence 'the king of America is bald' is false. We assert '$\sim\sim Bk$' because '$\sim Bk$' seems to entail that the king of America has hair, and that claim must be false, too.

Russell showed how to regiment the sentence as a definite description, so that the paradox disappears. 'The king of America is bald' becomes 'there is thing which is a unique king of America and that thing is bald'. 'The king of America is not bald' becomes 'there is thing which is a unique king of America and that thing is not bald'. Conjoining their negations, as we did in KO, leads to no contradiction. We just deny the existence of a unique king of America.

In order to use Russell's technique on 'Pegasus' we have to interpret the name as a definite description. Quine introduces the predicate 'pegasizes' which stands for a property which holds of all and only things that have the properties that Pegasus does. I used this technique, of regimenting natural-language names as predicates, in the right sides of KEr, KPr, and GE. We can regiment NP, then, as NPr2, adopting Kant's claim that existence is not a predicate.

NPr2 $\quad\quad$ $\sim(\exists x)Px$

NPr2 is just the awkward claim NP written in first-order logic. Quine further thinks that we have solved a problem, that we no longer have any temptation to think that there is a Pegasus in order to claim '$\sim(\exists x)Px$'. A name can be meaningful, even if it has no bearer.

The distinction between the meaning of an expression, what some philosophers call its sense, and its reference derives from Frege. Frege used the example of the morning star (classically known as 'Phosphorus') and the evening star ('Hesperus') which both turned out to be the planet Venus. The terms 'Hesperus' and 'Phosphorus' referred to the same thing despite having different meanings. Similarly, 'Clark Kent' and 'Superman' refer to the same person while having different connotations.

To defend his claim that we can have meaningful terms without referents, that we can use terms like 'Pegasus' without committing to the existence of something named by 'Pegasus', Quine appeals to his method of determining our commitments by looking at interpretations of first-order logic, at the formal semantics. We call an interpretation on which all of a set of sentences come out true a model of that set. A logically valid formula is one which is true on every interpretation. When Quine says that to be is to be the value of a variable, he means that when we interpret our formal best theory, we need certain objects to model our theories. Only certain kinds of objects will model the theory. The objects which appear in a model of the theory are said, by that theory on that interpretation, to exist.

The formal system we base on the language **F** is sound and complete. Soundness means that every provable formula is true under all interpretations. Completeness means that any formula which is true under all interpretations is provable. The formulas which are true under all interpretations are the tautologies, or logical truths.

If we add non-logical axioms, we create a first-order theory of whatever those axioms concern.

If we add mathematical axioms, we can create a first-order mathematical theory. If we add axioms of physics, we can create a first-order physical theory. By adding axioms of our best scientific theories, we can, theoretically, construct a grand theory of everything. What our best theory says exists will be the objects in the domain of quantification of that theory. Unfortunately, the addition of non-logical axioms strong enough to do the work that scientists require turns our formal system incomplete. But, the view of the language of first-order logic as canonical persists among many philosophers. Our best expressions of our ontological commitments will be made in a canonical language, perhaps that of first-order logic. To be is to be the value of a variable of that language.

4.8.4. Applying Quine's Method

Quine's method for determining the ontological commitments of a theory can be applied to all sorts of questions. Consider again Quine's original worry about Pegasus. While names seem unavoidably referential, Quine urges us to avoid them as the sources of reference. Instead, we look to the domain of quantification, and the objects which serve as values of our variables. We regiment our best theory. It will include, or entail, a sentence like NPr2 which is logically equivalent to NPr3.

NPr2 $\quad\quad$ $\sim(\exists x)Px$
NPr3 $\quad\quad$ $(\forall x)\sim Px$

If we want to know whether NPr3 is true, we look inside the domain of quantification. If there is no object with the property of being Pegasus, we call this sentence true in the interpretation. We construct our best theory so that everything in the world is in our domain of quantification and nothing else is.

Universals are among the entities whose existence philosophers debate. Consider, as Quine does, redness. Is redness a thing beyond the particular things, like cardinals, that are red? We can use it grammatically as an object, as in RR.

RR $\quad\quad$ Redness is prettier than brownness.

A grammatical interpretation of the sensible sentence RR reifies redness, takes it to be an object. A profligate ontologist might thus be led to believe that there are abstract objects in addition to the concrete objects which have their properties. There is appendicitis in addition to people and their appendixes. There is redness in addition to fire engines and apples.

Quine insists that just as we can have red fire engines without redness, we can have meaningful statements without meanings. If we again turn to Quine's method, we see a way to neatly express the question. We regiment properties (universals) as predicates. We interpret predicates as sets of objects in the domain. So, the predicate 'is red' is interpreted as the set of all red things. The predicate 'has appendicitis' is taken as the set of all things that have appendicitis. Quine's method demands sets, but not properties. There is a set of red things, but there is no redness.

The difference between sets and properties is that sets are extensional: they are determined exclusively by their members, objects in the domain. If two sets have the same members, they are the same set. In contrast, properties are not necessarily defined extensionally. The set of creatures with hearts and creatures with kidneys is extensionally equivalent, they are the same creatures. But, the property of having a heart is different (intensionally, in terms of meaning) from the property of having a kidney.

Standard interpretations of first-order logic are extensional. We interpret predicates as sets of objects in a domain. Thus, standard interpretations of first-order logic do not reify properties. Still, we can reify them, if we wish, by including them among the objects in the domain of the theory. Thus, first-order logic can maintain a neutrality about existence that makes it compelling as a canonical language for expressing our most considered ontological commitments.

Paper Topics

1. What is the ontological status of abstract objects, like numbers or appendicitis? How can we characterize the debate between nominalists and realists? How does Quine's method facilitate the debate? Discuss the role of contextual definition Quine mentions at the end of DE.

2. Are there universals? What is the relationship between the distinction between singular and general statements and the distinction between abstract and concrete terms. Does that relationship help us understand the problem of universals? How does Quine's criterion facilitate the debate? Why does Quine reject meanings, in OWTI, and how does the rejection of meanings relate to the problem of universals?

3. What is the problem of non-existence? Consider the solutions provided by McX and Wyman. How does Quine's approach differ? How does Quine's approach relate to Russell's theory of definite descriptions?

4. What is a name? What is the relationship between naming and quantification? Discuss Quine's dictum, that to be is to be the value of a variable.

Suggested Readings

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§9: Color Incompatibility
4.9.1. Wittgenstein's Logical Atomism

The question of the scope and importance of deductive logic is a perennial topic for philosophers of logic. Some philosophers see it as a severely limited discipline, governing only the most obvious and incontrovertible inferences. Others see it as the foundation for all human reasoning, a normative discipline prescribing the ways in which rational beings should think. The beliefs of most philosophers nowadays lie somewhere between these two extremes. So, a central question for philosophers concerns the importance of the formal definitions of logical consequence that are developed in the first three chapters of this book for human reasoning more widely: Why do we study logic?

Kant, in the late eighteenth century had seen logic as a closed and complete discipline. But Frege's new mathematical logic, developed a century later, was remarkably more powerful, especially in its use of quantifiers and its general treatment of relations. The new logic was so successful in unifying propositional logic with term logic and generalizing their study that some philosophers initially supported hopes of it being the canonical language of all of knowledge. Through the twentieth century, Quine was a proponent of that view, and some philosophers continue to believe that first-order logic is canonical.

An early and enthusiastic philosophical application of Frege's logic to broader philosophical purposes came from Ludwig Wittgenstein, in his 1919 *Tractatus Logico-Philosophicus*. Wittgenstein, had visited Frege in Germany as a young student and had studied with Bertrand Russell, before World War I, in Cambridge, England. He worked on the *Tractatus* during his subsequent time in the Austrian army during the war. That treatise was published after the war, with an introduction by Russell. Russell described Wittgenstein's *Tractatus* as the culmination of the enterprise of logical analysis begun by Frege.

According to the *Tractatus,* the world is a collection of independent atomic facts combined according to logical principles. If we could get clear about the correct logic, Wittgenstein argued, then we could have a complete, accurate picture of the world in our best, most austere language.

The *Tractatus* was highly influential in Europe between the two wars, as the foundation of logical empiricism, or logical positivism. A group of logical empiricist philosophers influenced by the *Tractatus*, including Rudolph Carnap, Otto Neurath, Moritz Schlick, Carl Gustav (Peter) Hempel, and Herbert Feigl, came to be known as the Vienna Circle. A less-influential group called the Berlin Circle was centered around physicist Hans Reichenbach. The young A.J. Ayer visited Vienna from England and wrote about the movement. Ayer's *Language, Truth, and Logic* became a primary source for logical empiricism for English-speaking philosophers.

One could easily spend an entire term studying the *Tractatus*, let alone logical empiricism. *The Tractatus* is obscure, when read directly, consisting of a series of numbered aphorisms. There are seven main propositions, and all but the seventh have sets of explanatory sub-propositions. Wittgenstein seeks the limits of language in distinguishing between what can and what can not be said.

> §7. Whereof one cannot speak, thereof one must be silent.

The project of distinguishing between what can and can not be said, or between what can and can not be thought, naturally meets a fundamental difficulty. If we want to distinguish between, say, the backyards of two people, we can draw a boundary line. We perceive both sides of the line, and see the landscape divided. This side belongs to the Majors; this other side belongs to the Teodoros. In contrast, attempts to draw a line between what is expressible in language and what is not expressible are essentially more problematic. What is outside of the scope of language is inaccessible to expression. What is outside the boundary of thought can not be thought. We can look at both sides of a fense. We can only talk about and think about one side of the boundaries of language and thought.

Still, Wittgenstein believed that we can at least try to get clear about how our language functions and what its limits are. If we can not describe what is outside the limits of language, at least we can bump up against the edges.

The *Tractatus* presents an atomistic picture theory of meaning on which language mirrors the world. The structure both of language and of the world is governed by logical rules, like those depicted in the truth tables. Indeed, Wittgenstein was the first to develop truth tables, in the *Tractatus*; see §4.31. The world, he alleges, is a collection of independent states of affairs. Suppose that I am standing to the right of you. We have, let's say, two atomic facts, my standing and your standing, and a logical relation , being to the right of, which holds between those facts. I could stand to the right of you, or to the left of you, or on the other side of the planet. All of my relations to you are independent of you.

> §1.2. The world divides into facts.
> §2.06. From the existence or non-existence of one state of affairs, it is impossible to infer the existence or non-existence of another (Wittgenstein, *Tractatus Logico-Philosophicus*).

On Wittgenstein's view, language consists of atomic statements of those facts, connected into more complex statements by logical principles. Language mirrors the world by providing a logical structure which is structurally equivalent, or isomorphic, to the structure of the world.

> §2.16. If a fact is to be a picture, it must have something in common with what it depicts.
> §2.17. What a picture must have in common with reality, in order to be able to depict it - correctly or incorrectly - in the way it does, is its pictorial form (ibid).

Since language and logic have the same form as the world, we can know about the fundamental structure of reality by examining the fundamental structures of language and logic.

Of course, we can not rely on the surface grammar of natural language to reflect the structure of the world. Natural language is sloppy, full of misleading metaphors and pragmatic shorthand. If we want a true representation of the world, we must seek a finer language, like Frege's mathematical logic. Recall Frege's claim that his *Begriffsschrift* is like a microscope on our language.

> I believe I can make the relationship of my *Begriffsschrift* to ordinary language clearest if I compare it to that of the microscope to the eye. The latter, due to the range of its applicability, due to the flexibility with which it is able to adapt to the most diverse circumstances, has a great superiority over the microscope. Considered as an optical instrument, it admittedly reveals many imperfections, which usually remain unnoticed only because of its intimate connection with mental life. But as soon as scientific purposes place great demands on sharpness of resolution, the eye turns out to be inadequate. The microscope, on the other hand, is perfectly suited for such purposes... (Frege, Preface to *Begriffsschrift*)

Wittgenstein believed that Frege's logic, therefore, is the precision tool that the picture theory requires to represent the atomic facts of the world, and to show how they are related and combined. The correct logic will mirror the structure of the world. The correct logic, therefore, is essential to a proper understanding the nature of reality.

To see how the demands for precision are manifested, notice that my example of an atomic fact, my standing to the right of you, is misleading. My standing in a place is not an atomic fact, it is a complex fact. Complex facts are those which are analyzable into more-fundamental facts. You and I are both complex, since we are divisible into smaller parts. Standing is also a complex, since it is divisible into more fundamental facts about the position of our bodies. The true analysis of the world involves analyzing such complexes into their simple, atomic components.

Atomic facts are the foundational elements for the *Tractatus*, akin to the postulates of Euclidean geometry, say, or to Descartes's *cogito*. Wittgenstein's goal, in the *Tractatus*, was a theory of the world that analyzed all of the myriad complexes into their atomic elements. Such a theory would present a veridical and secure picture of the world. If we got the atomic elements right and combined them into the correct logic, our theory of the world would mirror the world precisely. We would have the isomorphism between language and the world that we want.

Because of its method of analyzing complex propositions into elementary ones, the kind of philosophy that was developed by the early Wittgenstein, under the influence of Frege and Russell, was called analytic philosophy. The name 'analytic philosophy' remains as a characterization of Anglo-American philosophy, despite the lack of contemporary interest in the project of analysis in this sense. But Wittgenstein's original plan was to use the new logic, because of its utility for analysis, to represent the atomic facts of the world in elementary propositions and their logical combinations.

4.9.2. The Problem

The problems facing atomism and logical empiricism arise in the *Tractatus* already, in the worry about whether there really are independent atomic facts. Atomic facts are supposed to be the most basic, not analyzable into further simple facts. Facts about our bodies, we saw, are not atomic because they can be reduced to facts about parts of our bodies. Properties like standing are not atomic for the same reason. It is a challenge to try to think about what kinds of facts could be most fundamental, irreducible to other facts.

Wittgenstein never gives a clear example of an atomic fact. Russell used the example of the color of a spot in my field of vision. A dot in one's field of vision seems as likely a candidate for an atomic fact as any for severeal kinds of reasons. First, a small dot of color seems irreducible to other facts. Second, atomic facts are supposed, by definition, to be independent of each other. The color of one dot in my field of vision can be any color, independent of the color of any other spot in my field of vision. Last, however we construct our theories of the world, however complex we believe the world to be, the ultimate arbiter of those theories seems to be our sense experience, like the experience of the color of a spot in our field of vision. Differences in colors in our fields of vision allow us to read the scale of a thermometer, the position of the stars seen in a telescope, and the motion of an object travelling toward us. Related facts which also seem simple include auditory tones and odors and tastes.

But, since sight is, for most of us, the most fecund of the senses, let's stick to the color of a spot in our field of vision. Wittgenstein noticed that even such simple facts can not be atomic because they are not independent. Instead, they carry some sorts of entailment relations.

> §6.3751. It is clear that the logical product of two elementary propositions can neither be a tautology nor a contradiction. The statement that a point in the visual field has two different colors at the same time is a contradiction.

Spots in one's color field seem paradigmatically atomic. Atomic facts must be independent. But spots in our color field are not independent.

Jerrold Katz, in "The Problem in Twentieth-Century Philosophy," characterizes Wittgenstein's 6.3751 as the central problem in twentieth-century philosophy. To explicate the problem, he considers six sentences.

K1	The spot is red and blue.
K2	The spot is red.
K3	The spot is not blue.
K4	The spot has a color.
K5	Red is a color.
K6	The spot is green.

K2, K4, K5 and K6 are supposed to express atomic facts; K1 and K3 are supposed to be simple logical products of elementary propositions.

But K1 is a contradiction. K2 and K5 are incompatible, and K2 entails K3 and K4. There are substantial logical relations among these propositions even though they appear to be elementary. If such facts are not atomic, then it is hard to see how any facts could be atomic. The world appears not to be atomic in the way that the *Tractatus* depicts.

If the elementary propositions are inter-dependent, it is difficult to see how they could serve as the foundations of other beliefs. If the proposition that this spot is green entails that it is not red, and not purple, and that it is a color, and that spots are incompatible with each other, and so on, I can not just immediately and securely know a single, simple fact. Such claims would be comprehensible only *en masse*.

The problem of how to understand how elementary propositions can have logical relations among them has become known as the color incompatibility, or color exclusion, problem. As Katz observes, the problem is not merely about color.

> It is a general problem about the extralogical vocabulary of the language and about all the semantic properties and relations of the language (Katz 548).

The problem can be seen in any sentence whose truth seems to be both logical and depending on the meanings of terms. BU appears to be a special kind of sentence, one whose truth is guaranteed by its meaning, like a logical truth.

BU	Bachelors are single.

We can regiment BU into predicate logic.

BUr	$(\forall x)(Bx \supset Sx)$

But, the logic does not reveal the special status of BU. There are logical relations among the terms 'bachelor' and 'single'. But, the logic we have been studying does not show those relations. The atomists, including Wittgenstein and the logical empiricists who followed him, could not accommodate the relationship between various atomic facts in their logic.

4.9.3. Meaning Postulates

It might seem rather easy to treat the color incompatibility problem. We can just adopt statements like MP1 - MP3 as axioms.

MP1	All bachelors are unmarried.
MP2	Red is a color.
MP3	Red is not blue.

This proposal was explored by Rudolph Carnap, one of the more-prominent of the logical empiricists. Propositions like MP1 - MP3 are extra-logical; they are about meanings rather than about logic. Carnap's proposal is that we can stipulate whatever meaning relations we believe to be important.

The stipulation involved in adopting meaning postulates leads to two serious problems. First, we would have to adopt a lot of meaning postulates. Red is not blue, and not green, and not a ball of feta cheese, and not the Archbishop of Canterbury. It is not plausible that we believe in any conscious way all of the required meaning postulates.

Second, and more problematically, a long list of meaning postulates is not an explanation of why such postulates hold.

> Meaning postulates serve as constraints on the assignment of extensions to sentences, but they cannot explain the property common to the sentences they enumerate. Like Socrates's interlocutors, meaning postulates offer examples of the concept instead of the concept (Katz 553).

This second problem with meaning postulates is subtle, so let's take a moment to spell it out carefully. If I stipulate that no Ps or Qs are Rs, then it will follow that no Ps are Rs.

LT1 $\quad (\forall x)[(Px \lor Qx) \supset {\sim}Rx)] \supset (\forall x)(Px \supset {\sim}Rx)$

LT is a logical truth of **F**. But, LT holds for any values of P, Q, and R. LT does not tell us anything about the relationship between Ps and Rs. It does not tell us that there is a relationship between being a P that entails being an R. It says, for example, that if all blue or green things are not red then all blue things are not red. But, we want an explanation of the relationship between blue and not red. We want an explanation of the consequent of LT, not merely that it follows from its antecedent as a logical truth.

Compare LT1 with LT2, which is a logical truth of **PL**.

LT2 $\quad (P \supset Q) \supset [(Q \supset R) \supset (P \supset R)]$

LT2, like LT1, is not the result of any stipulation. It is a theorem of our logic. If meaning postulates were able to do the work that Carnap wants them to do, they would give the status that LT2 has, and that the entire LT1 has, to just the consequent of LT1.

Using meaning postulates to solve the color incompatibility problem makes sentences K1 - K6 true by stipulation. But we can stipulate anything we like. We can adopt scientific postulates about the world. We can also adopt axioms governing fictional worlds. Our use of logic within a system of postulates does not determine the truth of those postulates. We want the truth of propositions K1 - K6 to be true as a matter of the logic of the terms, like LT1 and LT2, rather than as a matter of stipulation, like the consequent of LT1.

4.9.4. Semantic Markers

In order to avoid the problems with Carnapian meaning postulates, Katz proposes a constraint on any solution of the color incompatibility problem.

> A new way out must reject Carnap's assumption that the external, logical structure of extralogical words is the source of analyticity, contradiction, and analytic entailment in connection with sentences like [K1 - K6]. It must assume instead that such properties and relations derive from the internal, sense structure of extralogical words (Katz 553).

Katz proposes that in addition to the mathematical logic of Frege, we need a formal theory of semantic entailment, one that gets to the analyticity of meanings. Just as we went beneath the level of the sentence moving from **PL** to **M**, we can move beneath the level of logical form to semantic form.

Katz calls the semantic structural properties of syntactically simple terms (like color terms) decompositional sense structure. Senses are meanings. Decompositional sense structure is not syntactic. It depends essentially on meanings, and not the forms of terms. A term like 'bachelor', which is syntactically simple, can be semantically complex.

> The sense of 'single man' is complex, being a compositional function of the senses of 'single' and 'man'. Since 'single man' and 'bachelor' have the same sense, the sense of 'bachelor' is complex (Katz 555).

Decompositional sense structure is not logical, as the color incompatibility problem shows. The consequent of LT1 is nothing like a logical truth. It is undeniably true on the given interpretation; green things can not be red. But, the non-redness of something, while derivable from its greenness, is not a logical entailment. It is a semantic entailment.

In order to formalize the notion of semantic entailment, Katz introduces a technical device he calls semantic markers. Semantic markers allow us to analyze concepts, like of being particular color, in such a way as to reveal the entailments like the ones expressed in K1 - K6. I will not pursue the complex details of Katz's device, here.

Katz uses semantic markers to represent the decompositional sense structure of what appeared to Wittgenstein to be elementary propositions 'This spot is blue' is not a semantically elementary proposition; it presupposes a variety of analytic entailments. On Katz's analysis, blueness can still be a primitive sense, in that it is not definable in terms of other senses. But, the primitiveness of the sense does not entail that it is semantically simple. It has analytic relations with other senses, despite being primitive. Katz calls the senses of basic color terms complex primitive senses. They are primitive in that they are not reducible to other senses. They are complex, since they have semantic relations to other senses. Senses are thus both inside and outside of logic. Sense entailments are additional to logical ones. But, they constrain logic, since they guide entailments.

> Since senses provide the fine-grained linguistic structure necessary for a model-theoretic explanation of why such sentences have such logical properties and relations, senses are inside logic in precisely Wittgenstein's sense of "hav[ing] an effect on one proposition's following from another" (Katz 572).

Katz's semantic markers have not caught on in the philosophical community at large. While they are patterned after Noam Chomsky's syntactic theories of language, they are much more contentious. Many philosophers are wary of meanings. Senses are objective, in that they transcend any particular thinker or language user. But they are not the kinds of things that we can perceive with our senses. Thus, some philosophers think of them as spooky entities. Still, senses give us a way of understanding the semantic relations among terms without abandoning Wittgenstein's atomism.

The more popular response to Wittgenstein's problem is holistic, abandoning the atomism of the logical empiricists and Katz's concept of semantic primitives. Many of the more prominent holists, like Quine, also deny the existence of meanings.

4.9.5. Logical Empiricism and Holism

Color incompatibility is a puzzle for both Wittgensteinian atomists and the logical empiricists that followed Wittgenstein because it looks as if there is a logical relationship between various atomic facts. To see how the problem manifests itself for logical empiricism, we need to look more closely at the broader aims of that philosophical movement.

The logical empiricists saw Wittgenstein's picture theory as accommodating a scientific view of the world. Scientific laws, for example, were mere generalization over, and reducible to, the separable atomic facts. The logical empiricists believed that all our legitimate claims could be traced to a core set of simple observations.

> There is a class of empirical propositions of which it is permissible to say that they can be verified conclusively. It is characteristic of these propositions, which I have elsewhere called "basic propositions," that they refer solely to the content of a single experience, and what may be said to verify them conclusively is the occurrence of the experience to which they uniquely refer...Propositions of this kind are "incorrigible,"...[in that] it is impossible to be mistaken about them except in a verbal sense (Ayer, *Language Truth and Logic*, p 10).

The logical empiricists claimed that all of science and philosophy could be founded on the basis of observation statements in conjunction with the logical and mathematical principles used to regiment and derive those observations. Claims that are not observable may be derived from the axiomatic observations, or introduced by definition. Lastly, some claims, like logical truths, are neither observable nor derivable from observable claims. Hume called such claims relations of ideas. The logical empiricists called them analytic truths. Among the analytic truths were supposed to be logical truths and, for logicists like Frege and Russell, the truths of mathematics. For the logical empiricists, all and only meaningful statements will be either analytic, observable, or derivable (using logic) from observable axioms.

A fundamental presupposition of logical empiricism, then, is that one can make a clear distinction between an observation statement and an analytic one. This distinction was rooted in Wittgenstein's distinction between sensible statements and logical nonsense. Let's take a moment to look at that distinction.

One of the most important advances in modern logic was its ability to characterize a broad, general concept of logical truth. Logical truths of **PL** are tautologies, complex statements which are true no matter the truth values of their component variables. Logical truths of **F** are true on any interpretation.

We might characterize logical truths as necessary truths. Descartes, for example, believed that the certainty of logic and mathematics provided essential support to his claim that our minds have substantial content built into their structures. From the claim that logic and mathematics are innate, it is reasonable to ask whether there are other innate ideas, including the idea of God.

Wittgenstein thought that characterizing logical truths as necessary imbues them with too much importance. In contrast, he called them nonsense. The only statements that can picture the world are those that have sense, that can be either true or false, that can picture accurately or not. Tautologies are empty of content.

§4.46. The proposition shows what it says, the tautology and the contradiction that they say nothing. The tautology has no truth conditions, for it is unconditionally true; and the contradiction is on no condition true. Tautology and contradiction are without sense.

§6.1251. Hence, there can *never* be surprises in logic.

Logical truths are unknowable because they are too thin to be objects of knowledge. They don't picture any fact. Wittgenstein wanted carefully to circumscribe what we can know.

The logical truths were, for Wittgenstein, logical nonsense. The logical empiricists called them merely analytic. All agreed that they were easily derivable within formal logic. Analytic truths were sharply contrasted with synthetic ones, which had to trace back, or reduce, in some way, to observation. Indeed, the whole of the atomist movement, from Locke and Hume through Wittgenstein and the logical empiricists rests on this distinction between analytic and synthetic propositions.

Quine attacked the logical empiricist's distinction between analytic and synthetic statements by arguing for holism. Holism is the denial of atomism. The holist claims that there are no individual statements which independent of larger theories. Just as the color facts, K1 - K6, were not independent, all claims are inter-related.

> Our statements about the external world face the tribunal of sense experience not individually but only as a corporate body ("Two Dogmas of Empiricism" 41).

Where the atomist like Wittgenstein applies Frege's logic to atomic, elementary propositions, the holist despairs of finding any simple facts. The holist denies that there is any real difference between analytic and synthetic claims, between truths of logic and empirical truths.

> It is obvious that truth in general depends on both language and extralinguistic fact. The statement "Brutus killed Caesar" would be false if the world had been different in certain ways, but it would also be false if the word "killed" happened rather to have the sense of "begat." Hence, the temptation to suppose in general that the truth of a statement is somehow analyzable into a linguistic component and a factual component. Given this supposition, it next seems reasonable that in some statements the factual component should be null; and these are the analytic statements. But, for all it's a priori reasonableness, a boundary between analytic and synthetic statements simply has not been drawn (Quine, "Two Dogmas of Empiricism," 70).

Our knowledge of synthetic propositions is supposed to be rooted in our sense experience of particular facts. But the particular beliefs that are supposed to be the starting points of our knowledge, the foundations, seem not to be independent. That is a lesson of the color incompatibility problem. Knowledge of purportedly-atomic facts seems to require, or presuppose, the understanding of a whole battery of other facts that come along with them. Knowledge that this spot is green entails knowledge that green is a color, that this spot is not red, and so on. This problem seems to undermine the claim that any atomic fact is given, as a foundational belief. If the basic facts are interconnected, they could not possibly be immediately perceivable. They would only be comprehensible as whole systems of claims, a larger theory, a corporate body.

This problem with the analytic/synthetic distinction, call it the holistic insight, is related to the inter-connectedness of individual statements we saw in the color incompatibility problem. Individual statements depend for their truth on a broader theory, in contrast to Wittgenstein's atomism. Hempel, another prominent logical empiricist, applied the holistic insight to his account of scientific reasoning.

> In the language of science, and for similar reasons even in prescientific discourse, a single statement usually has no experiential implications. A single sentence in a scientific theory does not, as a rule, entail any observations sentences; consequences asserting the occurrence of certain observable phenomena can be derived from it only by conjoining it with a set of other, subsidiary, hypotheses (Hempel, "Empiricist Criteria of Cognitive Significance: Problems and Changes," 56).

Wittgenstein and the logical empiricists presented a system on which individual sentences, pictures of states of affairs, were verified or disconfirmed on their own. Then, they could be connected by logic into a larger theory. The holist's claim is that the meaning of a single expression is elliptical, incomplete on its own. It requires, for its meaning, reference to an entire linguistic framework, a theoretical context which forms the background to that expression.

> If...cognitive significance can be attributed to anything, then only to entire theoretical systems formulated in a language with a well-determined structure (Hempel, "Empiricist Criteria of Cognitive Significance: Problems and Changes," 57).

Hempel here alludes to what has come to be known as semantic holism: the unit of empirical significance is not the individual sentence, but the entire theory. Holism comes in a variety of forms. Most strong, semantic holism claims that the meaning of any term or sentence depends on the meanings of all of our sentences. Meaning is a property of an entire language, not of individual terms. Less contentiously, confirmation holism claims that individual sentences are confirmed or refuted only by whole theories, not individually. Confirmation holism is a logical fact about sets of sentences. Even two contradictory sentences are compatible in the absence of a larger theory which prohibits contradiction.

Quine holds both the stronger semantic holism and the less-contentious confirmation holism. Wilfrid Sellars argues that the holistic conclusion is not merely about colors, and observation reports of them.

> It follows, as a matter of simple logic, that one couldn't have observational knowledge of *any* fact unless one knew many *other* things as well (Sellars, "Does Empirical Knowledge Have a Foundation?" 123).

If holism, even in its weak form, is correct, then the presupposition of atomism that some of our beliefs can serve as unassailable foundations for the rest of our beliefs is false. Holist criticisms undermine any given-ness of our purportedly basic beliefs. Given the constraints on knowledge, we could not know any particular fact unless we already knew a broader swath of background facts. We could not know that a spot is green unless we already knew that green is a color, that a spot which is green is not red, and so on.

> One couldn't form the concept of *being green*, and, by parity of reasoning, of the other colors, unless he already had them (Sellars, "Does Empirical Knowledge Have a Foundation?" 120).

If knowing that this spot is green requires prior knowledge of a larger background theory, it becomes difficult to see how one could come to know anything at all. The holist, then, has a strong critical argument against the atomist, but creates what seems to be an even-more intractable problem.

4.9.6. Summary

We have looked at two different kinds of responses to the color incompatibility problem. Carnap and Katz attempt to save atomism by exploring the logic of semantic entailments. Given first-order logic, there is no formal representation of the connections among K1 - K6. But, we can extend our logic so that there is a formal representation of those entailments.

In contrast, holists like Quine, Sellars, and Hempel give up the belief that there are elementary propositions. Quine, indeed, gives up on the idea that there are senses. Quine denies that there are any logical connections among K1 - K6. Instead, he believes that the connections are loose, at best causal connections.

These topics are far too broad to be considered in proper depth, here. We have reached the edge of logic and breached the barrier to the philosophy of language.

Paper Topics

1. The logical empiricists were epistemic foundationalists, seeking to explain all of human knowledge on the basis of some secure, fundamental beliefs. Some critics of foundationalism, inspired by Quinean holism, defend coherentism in epistemology. Compare the two kinds of epistemologies. Sosa, Sellars, Ayer and Quine would all be good readings.

2. In "Two Dogmas of Empiricism," Quine argues against the logical empiricist's reductionism. Evaluate Wittgenstein's project in light of Quine's criticisms. See Melchert for a good discussion of the *Tractatus*'s project, as well as Ayer.

3. Do meaning postulates solve the color incompatibility problem? See Carnap's article, as well as Quine's response in "Two Dogmas of Empiricism."

4. What are semantic markers? How do they attempt to solve the color incompatibility problem? In addition to the discussion in "The Problem...," see Katz's *Semantic Theory*.

5. How does the color incompatibility problem shift Wittgenstein away from his original project. Work through his "Remarks on Logical Form." See Allaire and/or Austin, as well.

6. What is the logical form of a sentence? Are there solutions, other than Carnap's, to the color incompatibility problem that rely on logical form? See the Pietroski article.

Suggested Readings

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§10: Second-Order Logic and Set Theory
4.10.1. Set Theory

At the end of Chapter 3, I introduced higher-order logics and noted that the language **S** was controversial. Many philosophers have argued that higher-order logics are not really logic. Perhaps most influentially, Quine calls second-order logic, "set theory in sheep's clothing" (*Philosophy of Logic*, p 66). Some philosophers, like Quine, take first-order logic with identity as a canonical language, the privileged language used for expressing one's most sincere beliefs and commitments. Many philosophers see the step from first-order logic to second-order logic as breaching a barrier.

As I mentioned in §3.11 when introducing the identity predicate, the line between logical and non-logical claims is not always clear or obvious. Most people take identity to be a logical relation. Most people take set theory to be mathematical. But, the difference between first-order logic and some versions of set theory is mainly just the inclusion of one symbol, ε, used for set-inclusion, and a few basic principles which govern that relation. These principles, the axioms of set theory, are very powerful. But, they are neatly presentable iin a compact form.

There are a wide variety of formulations of basic set theory. Some of these formulations differ in their consequences. There is dispute among mathematicians over which basic set theory is correct. There is dispute over whether there is a correct set theory. And, beyond basic set theory, there are lots of controversial extensions. These topics are for another place. Our interest in set theory is mainly just to consider the question of whether higher-order logics are logical, as many neo-Fregeans believe, or mathematical, as Quine believes.

For the purposes of our discussion, then, we can consider one simple set of axioms of set theory, which is standardly called ZF.

Zermelo-Fraenkel Set Theory[5]

Substitutivity:	$(\forall x)(\forall y)(\forall z)[y{=}z \supset (y{\in}x \equiv z{\in}x)]$
Pairing:	$(\forall x)(\forall y)(\exists z)(\forall w)[w{\in}z \equiv (w = x \lor w = y)]$
Null Set:	$(\exists x)(\forall y){\sim}x{\in}y$
Sum Set:	$(\forall x)(\exists y)(\forall z)[z{\in}y \equiv (\exists w)(z{\in}w \bullet w{\in}x)]$
Power Set:	$(\forall x)(\exists y)(\forall z)[z{\in}y \equiv (\forall w)(w{\in}z \supset w{\in}x)]$
Selection:	$(\forall x)(\exists y)(\forall z)[z{\in}y \equiv (z{\in}x \bullet \mathscr{F}u)]$, for any formula $\mathscr{F}$ not containing y as a free variable
Infinity:	$(\exists x)(a{\in}x \bullet (\forall y)(y{\in}x \supset Sy{\in}x)$

Note that In addition to ε, the axiom of infinity uses 'a' for the empty set, whose existence is guaranteed by the null set axiom, and 'S' for the function, '$y \cup \{y\}$', the definitions for the components of which are standard. 'S' is a successor function, essential to mathematics. In arithmetic, the successor function is used to generate the natural numbers. In ZF, we use it to generate an infinite set of sets.

Almost all of what we consider to be mathematics is derivable, with the addition of further definitions, from the axioms of ZF. Let's take a moment to sketch how the powerful tools of the real numbers can be constructed out of set theory. The discussion in this section will get just a little bit technical, but only the general form of the sketch is most important. You can easily skip to the next subsection on logicism.

First, we can define the natural numbers within set theory using any of various standard constructions, like those of Zermelo or Von Neumann. (Remember, 'a' stands for the empty set.)

[5] The presentation here follows Mendelson 1997, but with adjustments.

Zermelo:

0 = a
1 = {a}
2 = {{a}}
3 = {{{a}}}
...

Von Neumann

0 = a
1 = {a}
2 = {a, {a}}
3 = {a, {a}, {a, {a}}}
...

Using the Peano axioms (see 3.13.16) and the notion of an ordered pair, we can define standard arithmetic operations, like addition and multiplication. We can define the integers, **Z**, in terms of the natural numbers by using subtraction. Since -3 is 5-8, we can define -3 as the ordered pair <5,8>. But -3 could also be defined as <17,20>. To avoid ambiguity, we take the negative numbers to be equivalence classes of such ordered pairs. The equivalence class for subtraction is defined using addition: <a,b> ~ <c,d> iff a + d = b + c, where <a,b> ~ <c,d> indicates that <a,b> is in the same equivalence class as <c,d>. So, we can define **Z** = ...-3, -2, -1, 0, 1, 2, 3... in terms of $\mathbb{N}$, addition, and the notion of an ordered pair.

The rationals, **Q**, can be defined in terms of the integers, **Z**, by using ordered pairs of integers. a/b::<a,b>, where '<a,b> ~ <c,d> iff ad = bc' is the identity clause. The real numbers, $\mathbb{R}$, are differentiated from the rationals by their continuity. Both the rationals and the reals are dense: between any two there is a third. The reals are also continuous. There are again a variety of ways to define continuity set-theoretically. In the nineteenth century, mathematicians including Bolzano, Cauchy, and Weierstrass pursued the arithmetization of analysis. Part of their achievement was the epsilon-delta definition of continuous functions, due to Weierstrass in the 1860s, but based on ideas from Cauchy, 1821.

> a function f(x) is continuous at a if for any $\varepsilon > 0$ (no matter how small) there is a $\delta > 0$ such that
> for all x such that $|x - a| < \delta$, $|f(x) - f(a)| < \varepsilon$.

From this definition, we can find a definition of limits.

> A function f(x) has a limit L at a if for any $\varepsilon > 0$ there is a $\delta > 0$ such that
> for all x such that $|x - a| < \delta$, $|f(x) - f(a)| < \varepsilon$.

The ε-δ definition of limits is arithmetic. In addition to the arithmetic definitions of continuity and limits, Weierstrass, Dedekind, and Cantor pursued more rigorous definition of the reals, in terms of the rationals. Dedekind's derivation, from 1872, for example, relied on the concept a cut, which has become known as a Dedekind cut. The real numbers are identified with separations of the rationals, **Q**, into two sets, Q_1 and Q_2, such that every member of Q_1 is less than or equal to the real number and every member of Q_2 is greater. So, even though $\sqrt{2}$ is not a rational, it divides the rationals into two such sets. Not all cuts are produced by rational numbers. So, we can distinguish the continuity of the reals from the discontinuity of the rationals on the basis of these cuts. Real numbers are thus defined in terms of sets of rationals, the set of rationals below the cut. These sets have no largest member, since for any rational less than $\sqrt{2}$, for example, we can find another one larger. But, they do have an upper bound in the reals (i.e. the real number being defined).

By adding a definition of the real numbers in terms of the rational numbers to our definitions of the rationals in terms of the natural numbers, we have defined the reals in terms of the natural numbers. Such definitions do two things. First, they make it clear that infinitesimals and real numbers are defined using methods that human beings could naturally employ. They do not require controversial infinitistic inferences. Second, they make it plausible that we can reduce the problem of justifying our knowledge of mathematics to the problem of justifying our knowledge of just natural numbers. We have an ontological reduction of the objects of analysis to the objects of number theory: we need not assume the existence of any objects beyond the natural numbers in order to model analysis. And, we need not assume the existence of any objects beyond sets in order to model the natural numbers.

4.10.2. Logicism

The axioms of set theory are powerful indeed, strong enough to ground, in some sense, perhaps all of mathematics. The question facing us is whether they are logical. Is set theory a logical theory? Frege's grand project, the one for which his original *Begriffsschrift* was developed, is that mathematics is just logic in complicated disguise.

> Arithmetic...becomes simply a development of logic, and every proposition of arithmetic a law of logic, albeit a derivative one (Frege, *Grundlagen* §87).

Frege's claim is called logicism. Logicism is the intellectual heir of Leibniz's late-seventeenth/ early-eighteenth century proposal to reduce all propositions to elementary identities. Frege's argument for logicism mainly rests on his attempt to define the natural numbers merely by using logic. His project of reducing mathematics to logic thus requires two steps. One step is the reduction of the theory of natural numbers to logic. The other step of Frege's argument is the reduction of all of the rest of mathematics to the theory of natural numbers. In the last section, I sketched how mathematics is reducible to set theory. Frege thought that set theory was really just a logical theory.

> The present work will make clear that even an inference like that form n to $n + 1$, which on the face of it is peculiar to mathematics, is based on the general laws of logic, and that there is no need of special laws for aggregative thought (Frege, *Grundlagen* iv).

Set theory was a field in its infancy in Frege's time. Axiomatizations were only developed later, largely in response to problems arising from Frege's work. Definitions of numbers in terms of sets, like those of Zermelo and Von Neumann, above, were not yet available. So Frege defined the numbers using some basic principles which he took to be obvious, and thus logical. He established definitions of one-one correspondence, the property of having the same number as, number, zero, successor, and natural number. From these definitions, he derived definitions of each individual number. In the *Grundlagen*, he sketched these definitions and derivations. In the *Grundgesetze*, he developed the derivations fully.

The short explanation of Frege's definitions of numbers is that he took numbers to be certain kinds of sets, indeed sets of sets. More precisely, Frege took sets to be logical objects, extensions of predicates. The extension of a concept is the set of things which fall under that concept, or which have that property. Frege believed that the concept of the extension of a predicate is more precise than the concept of a set.

For one-one correspondence and the property of having the same number as, Frege relied on Hume's principle.

> We have to define the sense of the proposition "the number which belongs to the concept *F* is the same as that which belongs to the concept *G*"... In doing this, we shall be giving a general criterion for the identity of numbers. When we have thus acquired a means of arriving at a determinate number and of recognizing it again as the same, we can assign it a number word as its proper name. Hume long ago mentioned such a means: "When two numbers are so combined as that the one has always an unit answering to every unity of the other, we pronounce them equal" (Frege, *Grundlagen* §§62-3).

To define 'number', Frege relied directly on extensions of concepts.

> The number which belongs to the concept *F* is the extension of the concept "equal to the concept *F*" (Frege, *Grundlagen* §68).

Frege supposed that concepts are objects of thought. They are intermediary between objects and thoughts. When I think of a bluebird, there may or may not be any bluebird of which I am thinking. I can think of objects that don't exist, like the elephant standing on my head. I think of the concept of the elephant even in the absence of any elephant. So, concepts are not just elephants or bluebirds. Moreover, you and I may both think of bluebirds even though our particular thoughts are different. So, concepts are not our individual, subjective thoughts. They are objective but not physical.

Frege's definitions of number tell us when a number belongs to a concept. But numbers are objects themselves, not merely properties of concepts. Frege must provide a definition of the number terms without appealing merely to when they hold of concepts. To do, he argues that numbers are second-order extensions, extensions of extensions. In particular, numbers are extensions of all extensions of a particular size. In set-theoretic terms, they are sets of sets. Two is the set of all two-membered sets.

For Frege, 0 belongs to a concept if nothing falls under the concept. Thus, Frege can define zero by appealing to a concept with no extension.

> 0 is the Number which belongs to the concept "not identical with itself" (Frege, *Grundlagen* §74).

Again in set-theoretic terms, 0 is the set of all sets which are not identical to themselves (i.e. the number of x such that $x \neq x$). The definitions of the rest of the numbers can be generated inductively, using the successor definition.

> "There exists a concept *F*, and an object falling under it *x* such that the Number which belongs to the concept *F* is *n* and the Number which belongs to the concept 'falling under *F* but not identical with *x*' is *m*" is to mean the same as "*n* follows in the series of natural numbers directly after *m*" (Frege, *Grundlagen* §76.)

The number one applies to a concept if that concept a) applies to at least one thing; and b) if it applies to two things, they are the same thing. In our notation: $(\exists!x)Fx \rightleftharpoons (\exists x)[Fx \bullet (\forall y)(Fy \supset y{=}x)]$ The number one then may be defined as the number which belongs to the concept 'identical to zero', since there is only one concept zero. More succinctly, one is the set of all one-membered sets; two is the set of all two-membered sets.

Frege's logicist project, as he originally conceived it, was devastated by a paradox devised by Bertrand Russell. Russell sent word of the paradox to Frege just as the second volume of the *Grundgesetze* was being published. Frege added an attempt to avoid the paradox, but it was, in the end, unsuccessful. Russell worked out a more thorough, if not fully intuitive, way to avoid the paradox, and used it in his *Principia Mathematica*.

The source of Russell's paradox is Frege's use of an unrestricted axiom of comprehension.

Frege's logic can be called naive set theory, for its use of an axiom of comprehension (or abstraction). The axiom of comprehension says that any property determines a set. For Frege, the relevant version is that every predicate has an extension. Frege adds this comprehension claim to his treatment in the *Grundgesetze*, as Axiom 5.

Axiom 5 $\{x|Fx\} = \{x|Gx\} \equiv (\forall x)(Fx \equiv Gx)$

Axiom 5 leads to Proposition 91.

Proposition 91 $Fy \equiv y \in \{x|Fx\}$

Axiom 5 says that the extensions of two concepts are equal if and only if the same objects fall under the two concepts. In other words, the set of Fs and the set of Gs are identical iff all Fs are Gs. Proposition 91 says that a predicate F holds of a term iff the object to which the term refers is an element of the set of Fs. Both statements assert the existence of a set of objects which corresponds to any predicate, though this claim could be made more explicitly with a higher-order quantification.

To derive Russell's paradox, take F to be 'is not an element of itself'. So, x is not element of itself is expressed as 1.

1 $x \notin x$ (which is short for '$\sim x \in x$')

Now, take y to be the set of all sets that are not elements of themselves, as at 2.

2 $y = \{x|x \notin x\}$

Substitute '$\{x|x \notin x\}$' for y, and the property of not being an element of itself for F in Proposition 91. On the left, you get 3.

3 $\{x|x \notin x\} \notin \{x|x \notin x\}$

On the right side you get 4.

4 $\{x|x \notin x\} \in \{x|x \notin x\}$

Putting the two sides together, you get Russell's paradox, RP.

RP $\{x|x \notin x\} \notin \{x|x \notin x\} \equiv \{x|x \notin x\} \in \{x|x \notin x\}$

RP is of the form '$\sim P \equiv P$'. For those of you who like their contradictions in the form $\alpha \bullet \sim\alpha$, note the derivation RPC.

RPC

1.	$\sim P \equiv P$	
2.	$(\sim P \supset P) \bullet (P \supset \sim P)$	1, Equiv
3.	$\sim P \supset P$	2, Simp
4.	$P \supset \sim P$	2, Simp
5.	$\sim P \vee \sim P$	4, Impl
6.	$P \vee P$	3, Impl, DN
7.	$\sim P$	5, Tautology
8.	P	6, Tautology
9.	$P \bullet \sim P$	8, 7, Conjunction

QED

In current axiomatic set theory, like ZF, we avoid we avoid the paradoxes of unrestricted comprehension by building sets iteratively. We start with a few basic axioms, and build up the rest from those. We thus use what is called an iterative concept of set. We iterate, or list, the sets from the most simple.

Russell's solution to the paradoxes is to introduce a theory of types. According to the theory of types, a set can not be a member of itself. ZF provides a similar solution in that there is no way to generate the problematic sets. Frege was able to argue that mathematics reduced to logic because he claimed just the basic insight that every property determined a set. But both Russell's solution and ZF substitute a substantive set theory for Frege's original logical insight. Set theory does not appear to be a logical theory, but a mathematical theory. Thus, given the paradoxes, Frege was able to show that mathematics is reducible to mathematics (i.e. set theory), but not to logic.

4.10.3. Second-Order Logic

The question we are pursuing is whether second-order logic is logic or mathematics. We are taking set theory to be mathematics and **F**, for example, to be logic. When we interpret **F**, we specify a domain for the variables to range over. Sometimes we use restricted domains. If we want to interpret number theory, for example, we restrict our domain to the integers. If we want to interpret a biological theory, we might restrict our domain to species. For our most general reasoning, we take an unrestricted domain: the universe, everything there is. Consider BH.

BH There are blue hats. $(\exists x)(Bx. \bullet Hx)$

On standard semantics, for BH to be true there must exist a thing which will serve as the value of the variable 'x', and which has both the property of being a hat and being blue. As Quine says, to be is to be the value of a variable. Our most sincere commitments arise from examining the domain of quantification for our best theory of everything.

Now, consider a sentence of second-order logic, SP.

SP Some properties are shared by two people.
$(\exists X)(\exists x)(\exists y)(Px \bullet Py \bullet x \neq y \bullet Xx \bullet Xy)$

For SP to be true, there must exist two people, and there must exist a property. The value of the variable 'X' is not an ordinary object, but a property of an object. By quantifying over properties, we take properties as kinds of objects; we need some thing to serve as the value of the variable.

We could take the objects which serve as the values of predicate variables to be Platonic forms, or eternal ideas. But commitments to properties, in addition to the objects which have those properties, is metaphysically contentious. The first-order sentence about blue hats referred only to an object with properties. The second-order sentence reifies properties. Is there really blueness, in addition to blue things?

The least controversial way to understand properties is to take them to be sets of the objects which have those properties. We call this conception of properties extensional. On an extensional interpretation, 'blueness' refers to the collection of all blue things; the taller-than relation is just the set of ordered pairs all of whose first element is taller than its second element. Thus, second-order logic at least commits us to the existence of sets. Second-order logic, in its least-controversial interpretation, is some form of set theory.

We might want to include sets in our ontology if we think there are mathematical objects. But, we need not include them under the guise of second-order logic. We can instead take them to be values of first-order variables. We can count them as among the objects in the universe, in the domain of quantification, rather than sneaking them in through the interpretations of second-order variables. Quine's complaints about second-order logic, that it is set theory in sheep's clothing, are based on this sneakiness.

In favor of second-order logic, it is difficult to see how one could regiment sentences like many of those in §3.14 in first-order logic.

3.14.3 No two distinct things have all properties in common.
$(\forall x)(\forall y)[x \neq y \supset (\exists X)(Xx \bullet {\sim}Xy)]$

3.14.4 Identical objects share all properties.
$(\forall x)(\forall y)[x=y \supset (\forall Y)(Yx \equiv Yy)]$

3.14.9 Everything has some relation to itself.
$(\forall x)(\exists V)Vxx$

3.14.10 All people have some property in common.
$(\forall x)(\forall y)[(Px \bullet Py) \supset (\exists Y)(Yx \bullet Yy)]$

3.14.11 No two people have every property in common.
$(\forall x)(\forall y)[(Px \bullet Py) \supset (\exists Z)(Zx \bullet {\sim}Zy)$

3.14.15 Some relations are transitive.
$(\exists X)(\forall x)(\forall y)(\forall z)[(Xxy \bullet Xyz) \supset Xxz]$

3.14.16 Some relations are symmetric, while some are asymmetric.
$(\exists X)(\forall x)(\forall y)(Xxy \supset Xyx) \bullet (\exists X)(\forall x)(\forall y)(Xxy \supset {\sim}Xyx)$

The possibility of deriving the properties of identity from the second-order axioms, rather than introducing a special predicate with special inferential properties, is especially tempting.

3.14.17 $x=y \rightleftarrows (\forall X)(Xx \equiv Xy)$

Some philosophers favor using schematic predicate letters in lieu of predicate variables. With schematic letters, we regiment the law of the excluded middle, for example, as LEMS, rather than 3.14.6, with the understanding that any wff of **F** can be substituted for 'α'.

LEMS $\alpha \vee {\sim}\alpha$
3.14.6 $(\forall X)(X \vee {\sim}X)$

Schematic letters are metalinguistic variables. What those who favor schematic letters to second-order logic are really admitting is that we can not formulate claims like LEMS in our canonical language. We must, instead, ascend to a metalanguage, using metalinguistic variables.

So, it does seem that second-order logic is some form of set theory. Precisely what form of set theory it is depends on the semantics for the specific language of second-order logic we adopt. That is a topic for elsewhere. Given that second-order logic is some type of set theory, the question to ask is whether that is a problem for the language. Does its value in expressing some natural claims and their inferences outweigh its controversial ontological commitments? What exactly differentiates logic and mathematics? Is there a firm line between the disciplines? What is the purpose of logic? Is there one right logic?

This textbook, especially the first three chapters, covers themes which philosophers sometimes call baby logic. To try to answer any of the questions in the previous paragraph, one must explore the more-mature logics. This book is an introduction to a vast and burgeoning field, one which raises many important questions about human reasoning, ontology, epistemology, and other philosophical topics. More importantly, mastering the content should give you the tools to work on philosophy the way in which many contemporary philosophers do.

Paper Topics

1. Is there a correct logic? The debate over second-order logic is one aspect of a larger question of determining a canonical language, and whether there even is such a best language.

2. What differentiates logic and mathematics?

3. What is the purpose of logic?

Suggested Readings

Quine's objections to second-order logic are found in his *Philosophy of Logic*, among other places; he has good discussions of schematic letters in his *Methods of Logic*. Stewart Shapiro makes a compelling case against Quine and for second-order logic in his *Foundations without Foundationalism: A Case for Second-Order Logic*.

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Summary of Rules and Terms

Names of Languages

PL: Propositional Logic
M: Monadic (First-Order) Predicate Logic
F: Full (First-Order) Predicate Logic
FF: Full (First-Order) Predicate Logic with functors
S: Second-Order Predicate Logic

Basic Truth Tables

~	α
0	1
1	0

α	·	β
1	1	1
1	0	0
0	0	1
0	0	0

α	∨	β
1	1	1
1	1	0
0	1	1
0	0	0

α	⊃	β
1	1	1
1	0	0
0	1	1
0	1	0

α	≡	β
1	1	1
1	0	0
0	0	1
0	1	0

Rules of Inference

Modus Ponens (MP)
$\alpha \supset \beta$
$\alpha \quad / \ \beta$

Modus Tollens (MT)
$\alpha \supset \beta$
$\sim\beta \quad / \ \sim\alpha$

Disjunctive Syllogism (DS)
$\alpha \vee \beta$
$\sim\alpha \quad / \ \beta$

Hypothetical Syllogism (HS)
$\alpha \supset \beta$
$\beta \supset \gamma \quad / \ \alpha \supset \gamma$

Conjunction (Conj)
α
$\beta \quad / \ \alpha \cdot \beta$

Addition (Add)
$\alpha \quad / \ \alpha \vee \beta$

Simplification (Simp)
$\alpha \cdot \beta \quad / \ \alpha$

Constructive Dilemma (CD)
$(\alpha \supset \beta) \cdot (\gamma \supset \delta)$
$\alpha \vee \gamma \quad / \ \beta \vee \delta$

Rules of Equivalence

DeMorgan's Laws (DM)

$\sim(\alpha \cdot \beta) \rightleftarrows \sim\alpha \vee \sim\beta$

$\sim(\alpha \vee \beta) \rightleftarrows \sim\alpha \cdot \sim\beta$

Association (Assoc)

$\alpha \vee (\beta \vee \gamma) \rightleftarrows (\alpha \vee \beta) \vee \gamma$

$\alpha \cdot (\beta \cdot \gamma) \rightleftarrows (\alpha \cdot \beta) \cdot \gamma$

Distribution (Dist)

$\alpha \cdot (\beta \vee \gamma) \rightleftarrows (\alpha \cdot \beta) \vee (\alpha \cdot \gamma)$

$\alpha \vee (\beta \cdot \gamma) \rightleftarrows (\alpha \vee \beta) \cdot (\alpha \vee \gamma)$

Commutativity (Com)

$\alpha \vee \beta \rightleftarrows \beta \vee \alpha$

$\alpha \cdot \beta \rightleftarrows \beta \cdot \alpha$

Double Negation (DN)

$\alpha \rightleftarrows \sim\sim\alpha$

Contraposition (Cont)

$\alpha \supset \beta \rightleftarrows \sim\beta \supset \sim\alpha$

Material Implication (Impl)

$\alpha \supset \beta \rightleftarrows \sim\alpha \vee \beta$

Material Equivalence (Equiv)

$\alpha \equiv \beta \rightleftarrows (\alpha \supset \beta) \cdot (\beta \supset \alpha)$

$\alpha \equiv \beta \rightleftarrows (\alpha \cdot \beta) \vee (\sim\alpha \cdot \sim\beta)$

Exportation (Exp)

$\alpha \supset (\beta \supset \gamma) \rightleftarrows (\alpha \cdot \beta) \supset \gamma$

Tautology (Taut)

$\alpha \rightleftarrows \alpha \cdot \alpha$

$\alpha \rightleftarrows \alpha \vee \alpha$

Six Derived Rules for the Biconditional

Rules of Inference

Biconditional Modus Ponens (BMP)

$\alpha \equiv \beta$

$\alpha \qquad / \beta$

Biconditional Modus Tollens (BMT)

$\alpha \equiv \beta$

$\sim\alpha \qquad / \sim\beta$

Biconditional Hypothetical Syllogism
(BHS)

$\alpha \equiv \beta$

$\beta \equiv \gamma \qquad / \alpha \equiv \gamma$

Rules of Equivalence

Biconditional DeMorgan's Law (BDM)

$\sim(\alpha \equiv \beta) \rightleftarrows \sim\alpha \equiv \beta$

Biconditional Commutativity (BCom)

$\alpha \equiv \beta \rightleftarrows \beta \equiv \alpha$

Biconditional Contraposition (BCont)

$\alpha \equiv \beta \rightleftarrows \sim\alpha \equiv \sim\beta$

Rules for Quantifier Instantiation and Generalization

Universal Instantiation (UI)

$\dfrac{(\forall\alpha)\mathscr{F}\alpha}{\mathscr{F}\beta}$ for any variable α, any formula $\mathscr{F}$ containing α, and any singular term β

Universal Generalization (UG)

$\dfrac{\mathscr{F}\beta}{(\forall\alpha)\mathscr{F}\alpha}$ for any variable β, any formula $\mathscr{F}$ containing β, and for any variable α

Never UG within the scope of an assumption for conditional or indirect proof on a variable that is free in the first line of the assumption.
Never UG on a variable when there is a constant present, and the variable was free when the constant was introduced.

Existential Generalization (EG)

$\dfrac{\mathscr{F}\beta}{(\exists\alpha)\mathscr{F}\alpha}$ for any singular term β, any formula $\mathscr{F}$ containing β, and for any variable α

Existential Instantiation (EI)

$\dfrac{(\exists\alpha)\mathscr{F}\alpha}{\mathscr{F}\beta}$ for any variable α, any formula $\mathscr{F}$ containing α, and any *new* constant β

Quantifier Equivalence (QE)

$(\forall x)\mathscr{F}x \rightleftarrows {\sim}(\exists x){\sim}\mathscr{F}x$
$(\exists x)\mathscr{F}x \rightleftarrows {\sim}(\forall x){\sim}\mathscr{F}x$
$(\forall x){\sim}\mathscr{F}x \rightleftarrows {\sim}(\exists x)\mathscr{F}x$
$(\exists x){\sim}\mathscr{F}x \rightleftarrows {\sim}(\forall x)\mathscr{F}x$

Rules of Passage

For all variables α and all formulas Γ and Δ:

RP1:	$(\exists\alpha)(\Gamma \vee \Delta)$	$\rightleftarrows$	$(\exists\alpha)\Gamma \vee (\exists\alpha)\Delta$
RP2:	$(\forall\alpha)(\Gamma \bullet \Delta)$	$\rightleftarrows$	$(\forall\alpha)\Gamma \bullet (\forall\alpha)\Delta$

For all variables α, all formulas Γ containing α, and all formulas Δ not containing α:

RP3:	$(\exists\alpha)(\Delta \bullet \Gamma\alpha)$	$\rightleftarrows$	$\Delta \bullet (\exists\alpha)\Gamma\alpha$
RP4:	$(\forall\alpha)(\Delta \bullet \Gamma\alpha)$	$\rightleftarrows$	$\Delta \bullet (\forall\alpha)\Gamma\alpha$
RP5:	$(\exists\alpha)(\Delta \vee \Gamma\alpha)$	$\rightleftarrows$	$\Delta \vee (\exists\alpha)\Gamma\alpha$
RP6:	$(\forall\alpha)(\Delta \vee \Gamma\alpha)$	$\rightleftarrows$	$\Delta \vee (\forall\alpha)\Gamma\alpha$
RP7:	$(\exists\alpha)(\Delta \supset \Gamma\alpha)$	$\rightleftarrows$	$\Delta \supset (\exists\alpha)\Gamma\alpha$
RP8:	$(\forall\alpha)(\Delta \supset \Gamma\alpha)$	$\rightleftarrows$	$\Delta \supset (\forall\alpha)\Gamma\alpha$
RP9:	$(\exists\alpha)(\Gamma\alpha \supset \Delta)$	$\rightleftarrows$	$(\forall\alpha)\Gamma\alpha \supset \Delta$
RP10:	$(\forall\alpha)(\Gamma\alpha \supset \Delta)$	$\rightleftarrows$	$(\exists\alpha)\Gamma\alpha \supset \Delta$

Here are versions of each of the rules that are less-meta-linguistic and maybe easier to read:

RP1:	$(\exists x)(Px \vee Qx)$	$\rightleftarrows$	$(\exists x)Px \vee (\exists x)Qx$
RP2:	$(\forall x)(Px \bullet Qx)$	$\rightleftarrows$	$(\forall x)Px \bullet (\forall x)Qx$
RP3:	$(\exists x)(\mathscr{F} \bullet Px)$	$\rightleftarrows$	$\mathscr{F} \bullet (\exists x)Px$
RP4:	$(\forall x)(\mathscr{F} \bullet Px)$	$\rightleftarrows$	$\mathscr{F} \bullet (\forall x)Px$
RP5:	$(\exists x)(\mathscr{F} \vee Px)$	$\rightleftarrows$	$\mathscr{F} \vee (\exists x)Px$
RP6:	$(\forall x)(\mathscr{F} \vee Px)$	$\rightleftarrows$	$\mathscr{F} \vee (\forall x)Px$
RP7:	$(\exists x)(\mathscr{F} \supset Px)$	$\rightleftarrows$	$\mathscr{F} \supset (\exists x)Px$
RP8:	$(\forall x)(\mathscr{F} \supset Px)$	$\rightleftarrows$	$\mathscr{F} \supset (\forall x)Px$
RP9:	$(\exists x)(Px \supset \mathscr{F})$	$\rightleftarrows$	$(\forall x)Px \supset \mathscr{F}$
RP10:	$(\forall x)(Px \supset \mathscr{F})$	$\rightleftarrows$	$(\exists x)Px \supset \mathscr{F}$

Rules Governing the Identity Predicate (ID)

For any singular terms, α and β:

IDr. Reflexivity: $\alpha=\alpha$

IDs. Symmetry: $\alpha=\beta \rightleftarrows \beta=\alpha$

IDi. Indiscernibility of Identicals

$\mathscr{F}\alpha$
$\alpha=\beta \quad / \mathscr{F}\beta$

Solutions to Exercises

Chapter 1
Exercises 1.1

1. P1. Statements are meaningful if they are verifiable.
   P2. There are mountains on the other side of the moon.
   P3. No rocket has confirmed this, but we could verify it to be true.
   C. The original statement is significant.

2. P1. Everything in nature represents some state of mind.
   P2. This state of mind can be depicted by presenting its natural appearance as a picture.
   P3. An enraged man is a lion, a cunning man is a fox, a firm man is a rock, and a learned man is a torch.
   P4. Distances behind and in front of us are respectively images of memory and hope.
   C. It is not only words that are symbolic, but rather, it is things.

3. P1. As humans, we should believe in the theory that best accounts for our sense experience.
   P2. If we believe in a theory, we must believe in its ontological commitments.
   P3. The ontological commitments of any theory are the objects over which that theory first-order quantifies.
   P4. The theory which best accounts for our sense experience first order quantifies over mathematical objects.
   C. We should believe that mathematical objects exist.

4. P1. The workingman cannot afford to sustain the manliest relations to men.
   P2. His work would be minimized in the market.
   C. The workingman does not have time for true integrity on a daily basis.

5. P1. It is hard not to verify in our peers the same weakened intelligence due to emotions that we observe in our everyday patients.
   P2. The arrogance of our consciousness, which in general, belongs to the strongest defense mechanisms, blocks the unconscious complexes.
   C. It is difficult to convince people of the unconscious and in turn to teach them what their conscious knowledge contradicts.

6. P1. The passage from one stage to another may lead to long-continued different physical conditions in different regions.
   P2. These changes can be attributed to natural selection.
   C. The dominant species are the most diffused in their own country and make up the majority of the individuals, and often the most well marked varieties.

7. P1. All of psychology has gotten stuck in moral prejudices and fears.
   P2. No one has come close to understanding it as the development of the will to power.
   P3. However, if a person even begins to regard the affects of hatred, envy, covetousness, and the lust to rule as conditions of life and furthermore, as factors essential to the general economy of life, he will begin to get seasick.
   P4. At this point, he begins to lose himself, and sail over morality.
   C. Psychology becomes again the path to fundamental problems.

8. P1. Man has no choice about his capacity to feel that something is good or evil.
   P2. Man has no automatic knowledge and thus no automatic values.
   P3. His values are a product of either his thinking or his evasions.
   C. What he will consider good or evil depends on his standard of value.

9. P1. Mathematics succeeds as the language of science.
P2. There must be a reason for the success of mathematics as the language of science.
P3. No positions other than realism in mathematics provide a reason.
C. We must be realists about mathematics.

10. P1. The faster you go, the quicker you get to your destination.
P2. As you go faster, time itself becomes compressed.
P3. But it is not possible to go so fast that you get there before you started.
C. Local timelines are temporally ordered.

11. P1. The sphere is the most perfect shape, needing no joint and being a complete whole.
P2. A sphere is best suited to enclose and contain things.
P3. The sun, moon, plants, and starts are seen to be of this shape.
C. The universe is spherical.

12. P1. Fools are entirely devoid of the fear of death.
P2. They have no accusing consciences to make them fear it.
P3. They feel no shame, no solicitude, no envy, and no love.
P4. They are free from any imputation of the guilt of sin.
C. The happiest men are those whom the world calls fools.

13. P. It is impossible for someone to scatter his fears about the most important matters if he knows nothing about the universe but gives credit to myths.
C. Without the study of nature, there is no enjoyment of pure pleasure.

14. P1. If understanding is common to all mankind, then reason must also be common.
P2. he reason which governs conduct by commands and prohibitions is common to us.
C. Mankind is under one common law and so are fellow citizens.

15. P1. Rulers define 'justice' as simply making a profit from the people.
P2. Unjust men come off best in business.
P3. Just men refuse to bend the rules.
C. Just men get less and are despised by their own friends.

16. P1. We ought to take mathematical sentences at face value.
P2. If we take some sentences to be non-vacuously true, then we have to explain our access to mathematical objects.
P3. The only good account of access is the indispensability argument.
P4. The indispensability argument fails.
C. We must take non-vacuous mathematical sentences to be false.

17. P1. Labor was the first price, in that it yielded money that was paid for all things.
P2. It is difficult to ascertain the proportion between two quantities of labor.
P3. Every commodity is compared with other exchanged commodities rather than labor.
C. Most people better understand the quantity of a particular commodity, than the quantity of labor.

18. P1. Strength alone is not enough to make a man into a master.
P2. No man has natural authority over his fellows.
P3. Force creates no right.
C. Authority comes from only agreed conventions between men.

19. P1. Mathematics is defined as the indirect measurement of magnitude and the determination of magnitudes by each other.
    P2. Concrete mathematics aims to discover the equations of phenomena.
    P3. Abstract mathematics aims to educe results from equations.
    C. Concrete mathematics discovers results by experiment and abstract mathematics derives results from the discovered equations and obtains unknown quantities from known.

20. P. Many plants only bear fruit when they do not grow too tall.
    C. In the practical arts, the theoretical leaves and flowers must not be constructed to sprout too high, but kept near to experience, which is their proper soil.

21. P1. The greatest danger to liberty is the omnipotence of the majority.
    P2. A democratic power is never likely to perish for lack of strength or resources, but it may fall because of the misdirection of this strength and the abuse of resources.
    C. If liberty is lost, it will be due to an oppression of minorities, which may drive them to an appeal to arms.

22. P1. If there is an analytic/synthetic distinction, there must be a good explanation of synonymy.
    P2. The only ways to explain synonymy are by interchangeability *salva veritate* or definition.
    P3. Interchangeability cannot explain synonymy.
    P4. Definition presupposes synonymy.
    C. There is no distinction between analytic and synthetic claims.

23. P1. The object of religion is the same as that of philosophy; it is the internal verity itself in its objective existence.
    P2. Philosophy is not the wisdom of the world, but the knowledge of things which are not of this world.
    P3. It is not the knowledge of external mass, empirical life and existence, but of the eternal, of the nature of God, and all which flows from his nature.
    P4. This nature ought to manifest and develop itself.
    C. Philosophy in unfolding religion merely unfolds itself and in unfolding itself it unfolds religion.

24. P1. That the world is my idea is a truth valid for every living creature, though only man can contemplate it.
    P2. In doing so, he attains philosophical wisdom.
    P3. No truth is more absolutely certain than that all that exists for knowledge and therefore this world is only object in relation to subject, perception of a perceiver.
    C. The world is an idea.

25. P. Every art and every inquiry, and similarly every action and pursuit, is thought to aim at some good.
    C. The good has rightly been declared to be that a which all things aim.

26. P1. We should be committed to the entities hypothesized by the mathematics in question.
    P2. There exist genuine mathematical explanations of empirical phenomena.
    C. We should be committed to the theoretical posits hypothesized by these mathematical explanations.

27. P1. By ‘matter’ we are to understand an inert, senseless substance, in which extension, figure, and motion do actually subsist.
    P2. Extension, figure, and motion are only ideas existing in the mind.
    P3. An idea can be like nothing but another idea.
    P4. Neither they nor their archetypes can exist in an unperceiving substance.
    C. The very notion of what is called matter, or corporeal substance, involves a contradiction in it.

28. P1. Reading challenges a person more than any other task of the day.
P2. It requires the type of training that athletes undergo, and with the same life-long dedication.
P3. Books must be read as deliberately and reservedly as they were written.
C. To read well, as in, to read books in a true spirit, is a noble exercise.

29. P1. Love, friendship, respect, and admiration are the emotional responses of one man to virtues of another, the spiritual payment given in exchange for the personal, selfish, pleasure which one man derives from virtues of another.
P2. To love is to value.
C. The man who does not value himself cannot value anyone or anything.

30. P1. The only course open to one who wished to deduce all our knowledge from first principles would be to begin with *a priori* truths.
P2. An *a priori* truth is a tautology.
P3. From a set of tautologies alone, only further tautologies can be further deduced.
P4. It would be absurd to put forward a system of tautologies as constituting the whole truth about the universe.
C. We cannot deduce all our knowledge from first principles.

31. P1. Men, in the state of nature, must have reached some point when the obstacles maintaining their state exceed the ability of the individual.
P2. Then the human race must either perish or change.
P3. Men cannot create new forces, only unite and direct existing ones.
C. They can preserve themselves by only combining forces great enough to overcome resistance.

32. P1. Physics can be defined as the study of the laws which regulate the general properties of bodies regarded *en masse*.
P2. In observing physics, all senses are used.
P3. Mathematical analysis and experiments help with observation.
C. In the phenomena of physics man begins to modify natural phenomena.

33. P1. If there were two indiscernible individuals in our world then there must be another possible world in which those individuals are switched.
P2. God could have had no reason for choosing one of these worlds over the other.
P3. God must have a reason for acting as she does.
C. There are not two indiscernible individuals in our world.

34. P1. In aristocratic countries, great families have enormous privileges, which their pride rests on.
P2. They consider these privileges as a natural right ingrained in their being, and thus their feeling of superiority is a peaceful one.
P3. They have no reason to boast of the prerogatives which everyone grants to them without question.
C. When public affairs are directed by an aristocracy, the national pride takes a reserved, haughty and independent form.

35. P1. It must be some one impression, that gives rise to every real idea.
P2. Self or person is not any one impression, but that to which our several impressions and ideas are supposed to have a reference.
P3. If any impression gives rise to the idea of self, that impression must continue invariably the same through the whole course of our lives, since self is supposed to exist after that manner.
P4. There is no impression constant and invariable.
P5. Pain and pleasure, grief and joy, passions and sensations succeed each other and never all exist at the same time.
P6. It cannot, therefore, be from any of these impressions or from any other that the idea of self is derived.
C. There is no idea of the self.

36. P1. Every violent movement of the will, every emotion, directly agitates the body.
P2. This agitation interferes with the body's vital functions.
C. The body is the objectivity of the will.

37. P1. The work of the defensive forces of the ego prevents repressed desires from entering the conscious during waking life, and even during sleep.
P2. The dreamer knows just a little about the meaning of his dreams as the hysteric knows about the significance of his symptoms.
P3. The technique of psychoanalysis is the act of discovering through analysis, the relation between manifest and latent dream content.
C. The only way to treat these patients s through the technique of psychoanalysis.

38. P1. Either mathematical theorems refer to ideal objects or they refer to objects that we sense.
P2. If they refer to ideal objects, the radical empiricist cannot defend our knowledge of them, since we never sense such objects.
P3. If they refer to objects that we sense, they are false.
P4. For the radical empiricist, mathematical theorems are either unknowable or false.
C. The radical empiricist cannot justify any proof of a mathematical theorem.

39. P1. The sense or meaning of a term determines its reference.
P2. It is impossible for terms to differ in extension while having the same intension.
P3. Reference can vary without variation in thought.
P4. The senses of terms must be able to vary without variation in thought.
C. Our thoughts do not determine the meanings of our terms; meanings are not in the head.

40. P1. I have a clear and distinct understanding of my mind, independent of my body.
P2. I have a clear and distinct understanding of my body, independent of my mind.
P3. Whatever I can clearly and distinctly conceive of as separate, can be separated by God, and so are really distinct.
C. My mind is distinct from my body.

Exercises 1.2

1. Valid, sound
2. Valid, unsound
3. Invalid
4. Invalid
5. Valid, sound
6. Invalid
7. Valid, sound
8. Invalid
9. Invalid
10. Valid, sound
11. Valid, unsound
12. Valid, unsound
13. Invalid
14. Invalid
15. Valid, sound
16. Invalid
17. Valid, unsound
18. Valid, unsound
19. Invalid
20. Valid, unsound
21. Valid, unsound
22. Valid, sound
23. Invalid
24. Invalid
25. Valid, unsound
26. Valid, unsound
27. Invalid
28. Valid, sound
29. Valid, unsound
30. Valid, sound

Exercises 1.3a*

1. A
2. ~A
3. P • Z
4. P ∨ K
5. M • A
6. P ≡ (D • G)
7. L ∨ ~L
8. (F • L) • ~C
9. B ⊃ W
10. C ⊃ M
11. ~S ∨ J
12. E ≡ A
13. S ≡ (C ∨ L)
14. H ∨ R
15. P ⊃ (C • F)
16. (L • H) ∨ D
17. E ∨ (W • P)
18. (H ∨ L) ⊃ E
19. ~(M ∨ S)
20. C ⊃ (~R • H)

* Some alternate uses of letters are permissible.

Exercises 1.3b*

1. If Louisa teaches in a middle school, then she teaches either English or history.
2. Louisa teaches English, not history.
3. If Lousia teaches English, then Javier and Suneel do too.
4. If Louisa doesn't have a Master's degree then she neither teaches English nor history.
5. If neither Javier nor Suneel teach English, the Louisa teaches history.
6. If Marjorie is a philosophy professor who teaches logic then Jeremy majors in philosophy.
7. If Jeremy is a college student, then he majors in philosophy and psychology.
8. Either Jeremy doesn't major in both philosophy and psychology or he doesn't major in physics.
9. If Marjorie is not a philosophy professor who teaches logic, then Jeremy majors in either psychology or physics.

10. Jeremy majors in philosophy if and only if he is a college student and Marjorie is a philosophy professor.
11. If Carolina has a garden then she plants vegetables and flowers.
12. If Carolina has a garden and her plants grow, then deer eat the plants.
13. If Carolina has a garden and her plants grow and she sprays them with pesticide, then the deer do not eat the plants.
14. If Carolina plants either vegetables or flowers and her plants grow then either she sprays the plants with pesticides or deer eat the plants.
15. Carolina's plants don't grow if and only if she does not spray them with pesticide.

*Alternate formulations are permissible.

Exercises 1.4a

1. No
2. No
3. Yes, ⊃
4. Yes, ~
5. No
6. No
7. Yes, ⊃
8. No
9. Yes, ≡
10. Yes, ∨
11. No
12. Yes, ≡
13. No
14. Yes, ~
15. Yes, ⊃

Exercises 1.4b

1. C • (P ∨ R)
2. (M • F) ⊃ B
3. (O • T) ≡ ~R
4. (P ∨ C) ∨ I
5. C ⊃ [K • (P • I)]
6. (G • S) ⊃ (D • C)
7. (S • C) ∨ (P • D)
8. (S ⊃ W) • (C ⊃ B)
9. ~P • (O • T)
10. M ⊃ (P • W)
11. (P • A) • R
12. T ⊃ (D ≡ H)
13. (P ∨ D) ⊃ (A ∨ R)
14. ~D ≡ (~P • ~T)
15. (H ∨ T) • (A ∨ R)
16. (M • K) ∨ D
17. P ∨ (C • T)
18. E ≡ (M • T)
19. [S ⊃ (~P • ~C)] ∨ (R • ~D)
20. (T ∨ ~S) • C

Exercises 1.5a

1. False
2. False
3. True
4. True
5. False
6. True
7. True
8. False
9. True
10. False
11. False
12. True
13. False
14. True
15. True
16. True
17. False
18. True
19. True
20. False

Exercises 1.5b

1. False
2. True
3. False
4. False
5. Unknown
6. Unknown
7. True
8. True
9. True
10. False
11. True
12. Unknown
13. False
14. False
15. Unknown
16. False
17. Unknown
18. True
19. True
20. False

Exercises 1.5c

1. True
2. Unknown
3. True
4. True
5. True

6. True
7. Unknown
8. False
9. True
10. False

Exercises 1.6a

1.

A	⊃	~	A
1	**0**	0	1
0	**1**	1	0

2.

B	⊃	(~	B	⊃	B)
1	**1**	0	1	1	1
0	**1**	1	0	0	0

3.

(C	•	~	C)	⊃	C
1	0	0	1	**1**	1
0	0	1	0	**1**	0

4.

(D	∨	~	D)	≡	D
1	1	0	1	**1**	1
0	1	1	0	**0**	0

5.

~	E	⊃	F
0	1	**1**	1
0	1	**1**	0
1	0	**1**	1
1	0	**0**	0

6.

G	≡	~	H
1	**0**	0	1
1	**1**	1	0
0	**1**	0	1
0	**0**	1	0

7.

I	•	(J	∨	I)
1	**1**	1	1	1
1	**1**	0	1	1
0	**0**	1	1	0
0	**0**	0	0	0

8.

(K	≡	L)	⊃	L
1	1	1	**1**	1
1	0	0	**1**	0
0	0	1	**1**	1
0	1	0	**0**	0

9.

(M	•	N)	∨	~	M
1	1	1	**1**	0	1
1	0	0	**0**	0	1
0	0	1	**1**	1	0
0	0	0	**1**	1	0

10.

~	O	∨	(P	⊃	O)
0	1	**1**	1	1	1
0	1	**1**	0	1	1
1	0	**1**	1	0	0
1	0	**1**	0	1	0

11.

(Q	⊃	R)	≡	(R	⊃	Q)
1	1	1	**1**	1	1	1
1	0	0	**0**	0	1	1
0	1	1	**0**	1	0	0
0	1	0	**1**	0	1	0

12.

(S	∨	T)	•	~	(T	⊃	S)
1	1	1	**0**	0	1	1	1
1	1	0	**0**	0	0	1	1
0	1	1	**1**	1	1	0	0
0	0	0	**0**	0	0	1	0

13.

(U	•	~	V)	⊃	(V	∨	U)
1	0	0	1	**1**	1	1	1
1	1	1	0	**1**	0	1	1
0	0	0	1	**1**	1	1	0
0	0	1	0	**1**	0	0	0

14.

~	[(W	∨	X)	•	~	X]	⊃	W
1	1	1	1	0	0	1	**1**	1
0	1	1	0	1	1	0	**1**	1
1	0	1	1	0	0	1	**0**	0
1	0	0	0	0	1	0	**0**	0

15.

[(~	Y	•	Z)	⊃	Y]	∨	(Y	≡	Z)
0	1	0	1	1	1	**1**	1	1	1
0	1	0	0	1	1	**1**	1	0	0
1	0	1	1	0	0	**0**	0	0	1
1	0	0	0	1	0	**1**	0	1	0

16.

(A	≡	~	B)	⊃	[(B	∨	~	B)	•	A]
1	0	0	1	**1**	1	1	0	1	1	1
1	1	1	0	**1**	0	1	1	0	1	1
0	1	0	1	**0**	1	1	0	1	0	0
0	0	1	0	**1**	0	1	1	0	0	0

17.

(C	∨	D)	⊃	E
1	1	1	**1**	1
1	1	1	**0**	0
1	1	0	**1**	1
1	1	0	**0**	0
0	1	1	**1**	1
0	1	1	**1**	0
0	0	0	**1**	1
0	0	0	**1**	0

18.

(F	•	G)	≡	H
1	1	1	**1**	1
1	1	1	**0**	0
1	0	0	**0**	1
1	0	0	**1**	0
0	0	1	**0**	1
0	0	1	**1**	0
0	0	0	**0**	1
0	0	0	**1**	0

19.

~	(I	∨	J)	•	K
0	1	1	1	**0**	1
0	1	1	1	**0**	0
0	1	1	0	**0**	1
0	1	1	0	**0**	0
0	0	1	1	**0**	1
0	0	1	1	**0**	0
1	0	0	0	**1**	1
1	0	0	0	**0**	0

20.

[L	⊃	(M	∨	N)]	≡	L
1	1	1	1	1	**1**	1
1	1	1	1	0	**1**	1
1	1	0	1	1	**1**	1
1	0	0	0	0	**0**	1
0	1	1	1	1	**0**	0
0	1	1	1	0	**0**	0
0	1	0	1	1	**0**	0
0	1	0	0	0	**0**	0

21.

[~	O	•	(P	⊃	O)]	∨	Q
0	1	0	1	1	1	**1**	1
0	1	0	1	1	1	**0**	0
0	1	0	0	1	1	**1**	1
0	1	0	0	1	1	**0**	0
1	0	0	1	0	0	**1**	1
1	0	0	1	0	0	**0**	0
1	0	1	0	1	0	**1**	1
1	0	1	0	1	0	**1**	0

22.

(~	R	∨	S)	•	(~	T	⊃	R)
0	1	1	1	**1**	0	1	1	1
0	1	1	1	**1**	1	0	1	1
0	1	0	0	**0**	0	1	1	1
0	1	0	0	**0**	1	0	1	1
1	0	1	1	**1**	0	1	1	0
1	0	1	1	**0**	1	0	0	0
1	0	1	0	**1**	0	1	1	0
1	0	1	0	**0**	1	0	0	0

23.

[U	⊃	(V	⊃	W)]	•	(V	∨	W)
1	1	1	1	1	**1**	1	1	1
1	0	1	0	0	**0**	1	1	0
1	1	0	1	1	**1**	0	1	1
1	1	0	1	0	**0**	0	0	0
0	1	1	1	1	**1**	1	1	1
0	1	1	0	0	**1**	1	1	0
0	1	0	1	1	**1**	0	1	1
0	1	0	1	0	**0**	0	0	0

24.

[~	X	≡	(Y	•	Z)]	⊃	(X	∨	Z)
0	1	0	1	1	1	**1**	1	1	1
0	1	1	1	0	0	**1**	1	1	0
0	1	1	0	0	1	**1**	1	1	1
0	1	1	0	0	0	**1**	1	1	0
1	0	1	1	1	1	**1**	0	1	1
1	0	0	1	0	0	**1**	0	0	0
1	0	0	0	0	1	**1**	0	1	1
1	0	0	0	0	0	**1**	0	0	0

25.

(A	⊃	B)	∨	(C	≡	D)
1	1	1	**1**	1	1	1
1	1	1	**1**	1	0	0
1	1	1	**1**	0	0	1
1	1	1	**1**	0	1	0
1	0	0	**1**	1	1	1
1	0	0	**0**	1	0	0
1	0	0	**0**	0	0	1
1	0	0	**1**	0	1	0
0	1	1	**1**	1	1	1
0	1	1	**1**	1	0	0
0	1	1	**1**	0	0	1
0	1	1	**1**	0	1	0
0	1	0	**1**	1	1	1
0	1	0	**1**	1	0	0
0	1	0	**1**	0	0	1
0	1	0	**1**	0	1	0

26.

(E	•	~	F)	⊃	(G	∨	H)
1	0	0	1	**1**	1	1	1
1	0	0	1	**1**	1	1	0
1	0	0	1	**1**	0	1	1
1	0	0	1	**1**	0	0	0
1	1	1	0	**1**	1	1	1
1	1	1	0	**1**	1	1	0
1	1	1	0	**1**	0	1	1
1	1	1	0	**0**	0	0	0
0	0	0	1	**1**	1	1	1
0	0	0	1	**1**	1	1	0
0	0	0	1	**1**	0	1	1
0	0	0	1	**1**	0	0	0
0	0	1	0	**1**	1	1	1
0	0	1	0	**1**	1	1	0
0	0	1	0	**1**	0	1	1
0	0	1	0	**1**	0	0	0

27.

[I	⊃	(J	•	K)]	∨	(L	≡	I)
1	1	1	1	1	**1**	1	1	1
1	1	1	1	1	**1**	0	0	1
1	0	1	0	0	**1**	1	1	1
1	0	1	0	0	**0**	0	0	1
1	0	0	0	1	**1**	1	1	1
1	0	0	0	1	**0**	0	0	1
1	0	0	0	0	**1**	1	1	1
1	0	0	0	0	**0**	0	0	1
0	1	1	1	1	**1**	1	0	0
0	1	1	1	1	**1**	0	1	0
0	1	1	0	0	**1**	1	0	0
0	1	1	0	0	**1**	0	1	0
0	1	0	0	1	**1**	1	0	0
0	1	0	0	1	**1**	0	1	0
0	1	0	0	0	**1**	1	0	0
0	1	0	0	0	**1**	0	1	0

28.

[(~	M	•	N)	∨	(O	⊃	P)]	≡	M
0	1	0	1	1	1	1	1	**1**	1
0	1	0	1	0	1	0	0	**0**	1
0	1	0	1	1	0	1	1	**1**	1
0	1	0	1	1	0	1	0	**1**	1
0	1	0	0	1	1	1	1	**1**	1
0	1	0	0	0	1	0	0	**0**	1
0	1	0	0	1	0	1	1	**1**	1
0	1	0	0	1	0	1	0	**1**	1
1	0	1	1	1	1	1	1	**0**	0
1	0	1	1	1	1	0	0	**0**	0
1	0	1	1	1	0	1	1	**0**	0
1	0	1	1	1	0	1	0	**0**	0
1	0	0	0	1	1	1	1	**1**	0
1	0	0	0	0	1	0	0	**1**	0
1	0	0	0	1	0	1	1	**1**	0
1	0	0	0	1	0	1	0	**1**	0

Exercises 1.6b

1. Tautologous
2. Contradictory
3. Tautologous
4. Tautologous
5. Tautologous
6. Contingent
7. Tautologous
8. Contingent
9. Contingent
10. Contradictory
11. Contingent
12. Tautologous
13. Contradictory
14. Contradictory
15. Contradictory
16. Contradictory
17. Tautologous
18. Contingent
19. Tautologous
20. Tautologous
21. Contradictory
22. Contingent
23. Contingent
24. Contingent
25. Contradictory
26. Contradictory
27. Tautologous
28. Contingent
29. Contingent
30. Contingent

Exercises 1.6c

1. Consistent
2. Consistent
3. Consistent
4. Contradictory
5. Contradictory
6. Inconsistent
7. Logically equivalent
8. Logically equivalent
9. Inconsistent
10. Contradictory
11. Logically equivalent
12. Logically equivalent
13. Consistent
14. Inconsistent
15. Inconsistent
16. Logically equivalent
17. Inconsistent
18. Inconsistent
19. Inconsistent
20. Consistent
21. Consistent
22. Logically equivalent
23. Contradictory
24. Logically equivalent
25. Consistent
26. Inconsistent
27. Consistent
28. Contradictory
29. Logically equivalent
30. Consistent
31. Contradictory
32. Logically equivalent
33. Contradictory
34. Contradictory
35. Inconsistent

Exercises 1.7*

1. Invalid; counterexample when A is false
2. Valid
3. Invalid; counterexample when C is true and D is false
4. Invalid; counterexample when E is false and F is true
5. Valid
6. Valid
7. Valid
8. Invalid; counterexample when M is true and N is false
9. Invalid; counterexample when P is true and Q is either true or false
10. Invalid; counterexample when R is true, S is true, T is false
11. Invalid; counterexample when X is true, Y is false, Z is true
12. Invalid; counterexample when A, B, and C are all false
13. Valid
14. Valid
15. Invalid; counterexample when J is true, K is false, L is true
16. Invalid; counterexample when M is true, N is false, O is either true or false, and P is true
17. Invalid; counterexample when Q is false, R is false, S is either true or fase, and T is true
18. Invalid; counterexample when W and X are false and Y and Z are true
19. Valid
20. Valid

* Alternative counterexamples are available for some of the invalid arguments.

Exercises 1.8a*

1. Invalid; counterexample when A is false, B is true, C is true, D is true, E is true
2. Invalid; counterexample when F is true, G is false, H is true, I is false, J is true
3. Invalid; counterexample when K is true, L is false, M is true, N is false, O is false
4. Invalid; counterexample when P is false, Q is true, R is false, S is true, and T is true
5. Invalid; counterexample when U is true, V is false, W is false, Xis true, Y is true, Z is true
6. Valid
7. Valid
8. Invalid; counterexample when L is false and J, K, M and N are all true
9. Valid
10. Invalid; counterexample when U is true, V is false, W is true
11. Valid
12. Invalid; counterexample when A is true, B is true, C is false, D is true, and E is false
13. Valid
14. Invalid; counterexample when K is false, L is false, M is false, N is false, O is false, and P is true
15. Invalid; counterexample when Q is false, and R, S, T and U are all true
16. Invalid; counterexample when W is false, and X, Y, and Z are all true
17. Invalid; counterexample when E is true, and A, B, C, D and F are all false
18. Invalid; counterexample when G is false, H is false, I is false, J is true, and K is true
19. Invalid; counterexample when O is true, and L, M, N, and P are all false
20. Valid
21. Invalid; counterexample when V is false, W is true, X is true, Y is false, and Z is false
22. Valid
23. Invalid; counterexample when G is true, H is false, I is false, J is true, and K is false
24. Invalid; counterexample when L is false, M is true, N is true, P is true, and Q is false
25. Invalid; counterexample when R is false, and S, T, U and V are all true
26. Valid
27. Invalid; counterexample when J is false, K is false, L is true, and M is true
28. Invalid; counterexample when N is either true or false, O, P, and Q are true
29. Valid
30. Invalid; counterexample when V is true, W is true, X is false, Y is false, and Z is true
31. Valid
32. Invalid; counterexample when R is true, S is false, T is true, U is false, and V is true.

* Alternative counterexamples are available for some of the invalid arguments.

Exercises 1.8b*

1. Consistent; a consistent valuation is when A is true, B is true, C is false, and D is true
2. Consistent; a consistent valuation is when A is false, B is true, C is false, D is true, E is true, F is true
3. Inconsistent
4. Inconsistent
5. Consistent; a consistent valuation is when A is false, B is false, C is true, D is true, E is false, F is true
6. Inconsistent
7. Inconsistent
8. Consistent; a consistent valuation is when A is true, B is true, C is true, D is false, E is true, F is false
9. Inconsistent
10. Consistent; a consistent valuation is when A is true, B is false, D is false, E is true, F is true
11. Consistent; a consistent valuation is when O is true, P is true, Q is false, R is true, S is true, and T is true
12. Consistent; a consistent valuation is when O is true, P is false, R is true, S is true, T is true
13. Consistent; a consistent valuation is when T is true and O, P, Q, R, and S are all false
14. Inconsistent
15. Consistent; a consistent valuation is when P is true, Q is false, R is false, S is true, and T is false
16. Consistent; a consistent valuation is when I is true, J is true, K is false, L is true, M is true, N is false
17. Consistent; a consistent valuation is when I is true, J is true, K is false, L is true, M is false, N is false
18. Inconsistent
19. Consistent; a consistent valuation is when I is true, J is false, K is false, L is true, M is false, N is true
20. Consistent; a consistent valuation is when I is true, J is true, K is false, L is true, M is true, N is false

* Alternative consistent valuations are available for some of the consistent sets of sentences.

Chapter 2

Exercises 2.1a

1.
1. V ⊃ (W ⊃ X)
2. V
3. ~X
4. W ⊃ X 1, 2 MP
5. ~W 3, 4 MT

QED

2.
1. X ⊃ Y
2. ~Y
3. X ∨ Z
4. ~X 1, 2 MT
5. Z 3, 4 DS

QED

3.
1. E ⊃ F
2. ~F
3. ~E ⊃ (G • H)
4. ~E 1, 2 MT
5. (G • H) 3, 4 MP

QED

4.
1. I ⊃ J
2. J ⊃ K
3. ~K
4. I ⊃ K 1, 2 HS
5. ~I 3, 4 MT

QED

5.
1. T ⊃ S
2. S ⊃ R
3. T
4. T ⊃ R 1, 2 HS
5. R 3, 4 MP

QED

6.
1. (I • L) ⊃ (K ∨ J)
2. I • L
3. ~K
4. K ∨ J 1, 2 MP
5. J 3, 4 DS

QED

7.
1. G ⊃ E
2. F ⊃ ~E
3. H ∨ F
4. ~H
5. F 3, 4 DS
6. ~E 2, 5 MP
7. ~G 1, 6 MT

QED

8.
1. ~Q ⊃ (N • O)
2. (N • O) ⊃ (P ⊃ Q)
3. M ∨ ~Q
4. ~M
5. ~Q 3, 4 DS
6. ~Q ⊃ (P ⊃ Q) 1, 2 HS
7. P ⊃ Q 6, 5, MP
8. ~P 7, 5, MT

QED

9.
1. A ⊃ D
2. D ⊃ (B ⊃ C)
3. B
4. A
5. A ⊃ (B ⊃ C) 1, 2 HS
6. B ⊃ C 4, 5 MP
7. C 3, 6 MP

QED

10.
1. L ∨ N
2. ~L
3. N ⊃ (M ∨ O)
4. (M ∨ O) ⊃ (P ≡ Q)
5. ~N 1, 2 DS
6. N ⊃ (P ≡ Q) 3, 4 HS
7. P ≡ Q 5, 6 MP

QED

11.
1. U ⊃ V
2. ~V
3. U ∨ W
4. W ⊃ X
5. ~U 1, 2 MT
6. W 3, 5 DS
7. X 4, 6 MP

QED

12.
1. X ⊃ Z
2. Z ⊃ Y
3. ~Y
4. ~X ⊃ A
5. X ⊃ Y 1, 2 HS
6. ~X 3, 5 MT
7. A 4, 6 MP

QED

13.
1. C ⊃ B
2. B ⊃ D

3. (C ⊃ D) ⊃ E
4. C ⊃ D 1, 2, HS
5. E 3, 4, MP
QED

14.
1. E ⊃ H
2. G ∨ ~F
3. ~G
4. H ⊃ F
5. E ⊃ F 1, 4 HS
6. ~F 2, 3 DS
7. ~E 5, 6 MT
QED

15.
1. J ⊃ L
2. L ⊃ (I • M)
3. (I • M) ⊃ K
4. ~K
5. J ⊃ (I • M) 1, 2 HS
6. J ⊃ K 3, 5 HS
7. ~J 4, 6 MT
QED

16.
1. N ∨ (Q ≡ R)
2. N ⊃ P
3. P ⊃ M
4. ~M
5. ~P 3, 4 MT
6. ~N 2, 5 MT
7. Q ≡ R 1, 6 DS
QED

17.
1. N ∨ (P • ~R)
2. (P •~R) ⊃ Q
3. N ⊃ O
4. ~O
5. ~N 3, 4, MT
6. P • ~R 1, 5, DS
7. Q 2, 6, MP
QED

18.
1. R ⊃ S
2. S ⊃ (T ∨ U)
3. R
4. ~T
5. R ⊃ (T ∨ U) 1, 2 HS
6. T ∨ U 3, 5 MP
7. U 4, 6 DS
QED

19.
1. Q ⊃ (~R ⊃ S)
2. T ∨ Q
3. ~T
4. R ⊃ T
5. Q 2, 3 DS
6. ~R ⊃ S 1, 5 MP
7. ~R 3, 4 MT
8. S 6, 7 MP
QED

20.
1. C ⊃ (D ≡ ~E)
2. (D ≡ ~E) ⊃ (B ∨ A)
3. C ⊃ ~B
4. C
5. C ⊃ (B ∨ A) 1, 2 HS
6. B ∨ A 4, 5 MP
7. ~B 3, 4 MP
8. A 6, 7 DS
QED

21.
1. ~J ⊃ K
2. K ⊃ (L ⊃ M)
3. J ⊃ M
4. ~M
5. ~J 3, 4 MT
6. K 1, 5 MP
7. L ⊃ M 2, 6 MP
8. ~ L 4, 7 MT
QED

22.
1. V ⊃ (W ∨ U)
2. X ∨ V
3. X ⊃ Y
4. ~Y
5. ~Y ⊃ ~W
6. ~X 3, 4 MT
7. V 2, 6 DS
8. W ∨ U 1, 7 MP
9. ~W 4, 5 MP
10. U 8, 9 DS
QED

23.
1. X ⊃ (Y ⊃ Z)
2. W ∨ X
3. W ⊃ Y
4. ~Y
5. ~W ⊃ Y
6. ~W 3, 4 MT
7. X 2, 6 DS
8. Y ⊃ Z 1, 7 MP
9. Y 5, 6 MP
10. Z 8, 9 MP
QED

24.
1. (H •~G) ⊃ F
2. F ⊃ (G ∨ J)
3. I ∨ (H • ~G)
4. I ⊃ G
5. ~G
6. (H • ~G) ⊃ (G ∨ J) 1, 2 HS
7. ~I 4, 5 MT
8. H • ~G 3, 7 DS
9. G ∨ J 6, 8 MP
10. I 9, 5, DS
QED

25.
1. A ⊃ B
2. B ⊃ (C ⊃ D)
3. E ∨ C
4. E ⊃ B
5. ~B
6. C ⊃ A
7. ~E 4, 5 MT
8. C 3, 7 DS
9. A 6, 8 MP
10. A ⊃ (C ⊃ D) 1, 2 HS
11. C ⊃ D 9, 10 MP
12. D 8, 11 MP
QED

Exercises 2.1b

1.
1. ~A ⊃ B
2. A ⊃ C
3. ~C / B
4. ~A 2,3 MT
5. B 1, 4 MP
QED

2.
1. D ⊃ E
2. E ⊃ F
3. D / F
4. D ⊃ F 1, 3 HS
5. F 3, 4 MP

3.
1. G ⊃ H
2. I ∨ G
3. ~I / H
4. G 2, 3 DS
5. H 1, 4 MP
QED

4.
1. J ⊃ K
2. K ⊃ L
3. ~L / ~J
4. J ⊃ L 1, 2 HS
5. ~J 3, 4 MT
QED

5.
1. Q ∨ R
2. Q ⊃ S
3. ~S / R
4. ~Q 2, 3 MT
5. R 1, 4 DS
QED

6.
1. M ⊃ N
2. N ⊃ (O ∨ P)
3. M
4. ~O / P
5. M ⊃ (O ∨ P) 1, 2, HS
6. O ∨ P 3, 5 MP
7. P 4, 6 DS
QED

7.
1. T ∨ U
2. T ⊃ V
3. U ⊃ W
4. ~V / W
5. ~T 2, 4 MT
6. U 1, 5 DS
7. W 3, 6 MP
QED

8.
1. F ⊃ G
2. G ⊃ E
3. H ∨ ~E
4. ~ H / ~F
5. ~E 3, 4 DS
6. F ⊃ E 1, 2 HS
7. ~F 5, 6 MT
QED

9.
1. ~C ⊃ D
2. D ⊃ A
3. C ⊃ B
4. ~B / A
5. ~C 3, 4 MT
6. ~C ⊃ A 1, 2 HS
7. A 5, 6 MP
QED

10.
1. X ⊃ Y
2. Y ⊃ Z
3. W ∨ X
4. W ⊃ Y
5. ~Y / Z
6. ~W 4, 5 MT
7. X 3, 6 DS
8. X ⊃ Z 1, 2 HS
9. Z 7, 8 MP

QED

Exercises 2.2a

1. MT
2. CD
3. HS
4. invalid
5. invalid
6. Add
7. invalid
8. MP
9. Conj
10. invalid
11. DS
12. Simp

Exercises 2.2b

1.
1. A ⊃ (C • D)
2. A • B
3. A 2, simp
4. C • D 1, 3 MP
5. C 4, simp

QED

2.
1. (M ⊃ N) • (O ⊃ P)
2. M • Q
3. M 2, simp
4. M ∨ O 3, add
5. N ∨ P 1, 3 CD

QED

3.
1. I ∨ J
2. ~I • K
3. ~I 2, simp
4. J 1, 3, DS
5. J ∨ L 4, add

QED

4.
1. (F ∨ G) ⊃ H
2. F • E
3. F 2, simp
4. F ∨ G 3, add
5. H 1, 4 MP

QED

5.
1. F ⊃ E
2. ~E • G
3. H
4. ~E 2, simp
5. ~F 1, 4 MT
6. ~F • H 3, 5 conj

QED

6.
1. (~A ⊃ B) • (C ⊃ D)
2. A ⊃ D
3. ~D
4. ~A 2,3 MT
5. ~A ∨ C 4, add
6. B ∨ D 1, 5 CD

QED

7.

1. W ⊃ X	
2. ~X • Y	
3. ~X	2, simp
4. ~W	1, 3 MT
5. ~W ∨ Z	4, add
6. (~W ∨ Z) • ~X	3, 5 conj

QED

8.

1. T ∨ S	
2. ~T	
3. U	
4. S	1, 2, DS
5. U • S	3, 4, Conj

QED

9.

1. (V • W) ⊃ X	
2. V • Y	
3. W • Z	
4. V	2, simp
5. W	3, simp
6. V • W	4, 5 conj
7. X	1, 6 MP

QED

10.

1. (E ∨ I) ⊃ H	
2. H ⊃ (F • G)	
3. E	
4. (E ∨ I) ⊃ (F • G)	1, 2 HS
5. E ∨ I	3, add
6. F • G	4, 5 MP
7. (F • G) • E	3, 6 conj

QED

11.

1. (J ⊃ L) • (K ⊃ M)	
2. J • M	
3. ~L	
4. J	2, simp
5. J ∨ K	4, add
6. L ∨ M	1, 5, CD
7. M	6, 3, DS

QED

12.

1. N ∨ ~~P	
2. ~N • Q	
3. ~P ∨ Q	
4. ~N	2, simp
5. ~~P	1, 4 DS
6. Q	3, 5 DS
7. ~~P • Q	5, 6 conj

QED

13.

1. M ⊃ N	
2. N ⊃ O	
3. M • P	
4. M	3, simp
5. M ⊃ O	1, 2 HS
6. O	4, 5 MP
7. O ∨ P	6, add

QED

14.

1. W ⊃ Z	
2. Z ⊃ (X ∨ Y)	
3. W • Y	
4. (X ⊃ U) • (Y ⊃ V)	
5. W	3, simp
6. W ⊃ (X ∨ Y)	1, 2 HS
7. X ∨ Y	5, 6 MP
8. U ∨ V	4, 7 CD

QED

15.

1. B ⊃ A	
2. ~A • D	
3. ~B ⊃ C	
4. ~A	2, simp
5. ~B	1, 4 MT
6. C	3, 5, MP
7. C ∨ A	6, Add

QED

16.

1. D ∨ E	
2. D ⊃ F	
3. ~F • G	
4. ~F	3, simp
5. ~D	2, 4 MT
6. E	1, 5 DS
7. E ∨ H	6, add
8. (E ∨ H) • ~F	4, 8 conj

QED

17.

1. R ⊃ S	
2. S ⊃ (T ⊃ U)	
3. R	
4. U ⊃ R	
5. R ⊃ (T ⊃ U)	1, 2 HS
6. T ⊃ U	5, 3, MP
7. T ⊃ R	6, 4, HS

QED

18.

1. (C ⊃ D) • (B ⊃ D)	
2. A • C	
3. A ⊃ C	
4. A	2, simp
5. C	3, 4, MP
6. C ∨ B	5, Add
7. D ∨ D`	1, 6, CD
QED	

19.

1. M ⊃ J	
2. (~M • ~J) ⊃ K	
3. ~J	
4. ~M	1, 3 MT
5. ~M • ~J	4, 3 conj
6. K	2, 5, MP
7. K ∨ N	7, add
QED	

20.

1. O ⊃ Q	
2. Q ⊃ P	
3. P ⊃ (R • S)	
4. O	
5. O ⊃ P	1, 2 HS
6. P	5, 4, MP
7. R • S	3, 6, MP
QED	

21.

1. (R ∨ T) ⊃ S	
2. S ⊃ U	
3. R	
4. (R ∨ T) ⊃ U	1, 2 HS
5. R ∨ T	3, Add
6. U	4, 5, MP
7. U ∨ T	6, Add
QED	

22.

1. I ⊃ J	
2. ~J • K	
3. ~J ⊃ L	
4. ~ ~I	
5. ~J	2, simp
6. L	3, MP
7. ~I	1, 5 MT
8. ~I ∨ K	7, add
9. K	4, 8 DS
10. K • L	6, 9 conj
QED	

23.

1. Q ⊃ R	
2. R ⊃ (S ∨ T)	
3. Q	
4. ~S • U	
5. Q ⊃ (S ∨ T)	1, 2 HS
6. S ∨ T	3, 5 MP
7. ~S	4, simp
8. T	6, 7 DS
9. T • Q	3, 8 conj
10. (T • Q) ∨ R	9, add
QED	

24.

1. (~V ⊃ W) • (X ⊃ Y)	
2. V ⊃ Z	
3. ~W • X	
4. ~Z • Y	
5. ~Z	4, simp
6. ~V	2,5 MT
7. ~V ∨ X	6, add
8. W ∨ Y	1, 7 CD
9. ~W	3, simp
10. Y	8, 9 DS
11. Y • ~V	6, 10 conj
QED	

25.

1. A ⊃ B	
2. B ⊃ (C ⊃ D)	
3. A • D	
4. ~D	
5. D • E	
6. A	3, simp
7. A ⊃ (C ⊃ D)	1, 2 HS
8. C ⊃ D	6, 7 MP
9. ~C	4, 8 MT
10. D	5, simp
11. D ∨ E	10, add
12. E	4, 11 DS
QED	

Exercises 2.2c

1.

1. $A \supset B$	
2. $\sim B \bullet C$	/ $\sim A \bullet \sim B$
3. $\sim B$	2, simp
4. $\sim A$	1, 3 MT
5. $\sim A \bullet \sim B$	3, 4 conj
QED	

2.

1. $(D \supset E) \bullet (F \supset G)$	
2. $H \supset D$	
3. $H \bullet G$	/ $E \vee G$
4. H	3, simp
5. D	2, 4 MP
6. $D \vee F$	5, add
7. $E \vee G$	1, 6 CD
QED	

3.

1. $I \supset J$	
2. $J \supset K$	
3. $I \bullet L$	/ $K \vee J$
4. $I \supset K$	1, 2 HS
5. I	3, simp
6. K	4, 5 MP
7. $K \vee J$	6, add
QED	

4.

1. $Q \supset \sim R$	
2. $\sim R \supset S$	
3. $Q \bullet \sim R$	/ $Q \bullet S$
4. $Q \supset S$	1, 2 HS
5. Q	3, simp
6. S	4, 5 MP
7. $Q \bullet S$	5, 6 conj
QED	

5.

1. $T \supset U$	
2. $V \vee \sim U$	
3. $\sim V$	/ $\sim T \vee W$
4. $\sim U$	2, 3 DS
5. $\sim T$	1, 4 MT
6. $\sim T \vee W$	5, add
QED	

6.

1. $E \vee F$	
2. $E \supset G$	
3. $\sim G \bullet H$	/ $F \vee H$
4. $\sim G$	3, simp
5. $\sim E$	2, 4 MT
6. F	1, 5 DS
7. $F \vee H$	6, add
QED	

7.

1. $M \supset N$	
2. $O \supset P$	
3. $M \vee O$	
4. $\sim N$	/ $P \vee M$
5. $\sim M$	1, 4 MT
6. O	3, 5 DS
7. P	2, 6 MP
8. $P \vee M$	7, add
QED	

8.

1. $A \supset B$	
2. $C \supset D$	
3. $\sim B$	
4. C	/ $\sim A \bullet D$
5. D	2, 4, MP
6. $\sim A$	1, 3, MT
7. $\sim A \bullet D$	6, 5, Conj
QED	

9.

1. $K \vee (L \bullet M)$	
2. $\sim K \bullet J$	
3. $L \supset J$	/ $J \vee M$
4. $\sim K$	2, simp
5. $L \bullet M$	1, 4 DS
6. L	5, simp
7. J	3, 6 MP
8. $J \vee M$	7, add
QED	

10.

1. $X \supset Y$	
2. $Y \supset Z$	
3. $W \vee X$	
4. $\sim W \bullet Y$	/ $Z \bullet \sim W$
5. $\sim W$	4, simp
6. X	3, 5 DS
7. $X \supset Z$	1, 2 HS
8. Z	6, 7 MP
9. $Z \bullet \sim W$	5, 8 conj
QED	

Exercises 2.3a

1.

1. A ⊃ B	
2. C • A	
3. A • C	2, com
4. A	3, simp
5. B	1, 4 MP

QED

2.

1. (A • B) ∨ (A • C)	
2. D ⊃ ~A	
3. A • (B ∨ C)	1, dist
4. A	3, simp
5. ~~A	4, DN
6. ~D	2, 5 MT

QED

3.

1. E ⊃ F	
2. ~~E • G	
3. ~~E	2, simp
4. E	3, DN
5. F	1, 4 MP

QED

4.

1. H ∨ J	
2. I • ~H	
3. ~H • I	2, com
4. ~H	3, simp
5. J	1, 4 DS

QED

5.

1. X ⊃ Y	
2. Z • ~Y	
3. ~Y • Z	2, com
4. ~Y	3, simp
5. ~X	1, 4 MT

QED

6.

1. F ⊃ (C ∨ D)	
2. ~[C ∨ (D ∨ E)]	
3. ~[(C ∨ D) ∨ E]	2, assoc
4. ~(C ∨ D) • ~E	3, DM
5. ~(C ∨ D)	4, simp
6. ~F	1, 5 MT

QED

7.

1. X ⊃ Y	
2. (~Y • Z) • T	
3. X ∨ W	
4. ~Y • (Z • T)	2, assoc
5. ~Y	4, simp
6. ~X	1, 5 MT
7. W	3, 6 DS

QED

8.

1. ~A ∨ B	
2. ~[(~A ∨ C) ∨ D]	
3. ~(~A ∨ C) • ~D	2, DM
4. ~(~A ∨ C)	3, simp
5. ~~A • ~C	4, DM
6. ~~A	5, simp
7. B	1, 6 DS

QED

9.

1. R • (S ∨ T)	
2. ~R ∨ ~S	
3. ~(R • S)	2, DM
4. (R • S) ∨ (R • T)	1, dist
5. R • T	3, 4 DS
6. T • R	5, com
7. T	6, simp

QED

10.

1. I • {~[J • (K ∨ L)] • M}	
2. (~J ∨ ~L) ⊃ N	
3. {I • ~[J • (K ∨ L)]} • M	1, assoc
4. I • ~[J•∨ (K ∨ L)]	3, simp
5. ~[J • (K ∨ L)] • I	4, com
6. ~[J • (K ∨ L)]	5, simp
7. ~[(J • K) ∨ (J • L)]	6, dist
8. ~(J • K) • ~(J • L)	7, DM
9. (~J ∨ ~K) • (~J ∨ ~L)	8, DM
10. (~J ∨ ~L) • (~J ∨ ~K)	9, com
11. ~J ∨ L	10, simp
12. N	2, 11 MP

QED

11.

1. J ⊃ K	
2. K ⊃ [L ∨ (M • N)]	
3. ~N • J	
4. J ⊃ [L ∨ (M • N)]	1, 2 HS
5. J • ~N	3, com
6. J	5, simp
7. L ∨ (M • N)	4, 6 MP
8. (L ∨ M) • (L ∨ N)	7, dist
9. (L ∨ N) • (L ∨ M)	8, com
10. L ∨ N	9, simp
11. ~N	3, simp
12. N ∨ L	10, com
13. L	11, 12 DS

QED

12.

1. ~[(G • H) • I]	
2. G • I	

3. ~[G • (H • I)}   1, assoc
4. ~G ∨ ~(H • I)   3, dist
5. G   2, simp
6. ~~G   5, DN
7. ~(H • I)   4, 6 DS
8. ~H ∨ ~I   7, DM
9. I • G   2, com
10. I   9, simp
11. ~~I   10, DN
12. ~I ∨ ~H   8, com
13. ~H   11, 12 DS

QED

13.
1. Q ⊃ R
2. ~(S ∨ T)
3. T ∨ Q
4. ~S • ~T   2, DM
5. ~T • ~S   4, com
6. ~T   5, simp
7. Q   3, 6 DS
8. R   1, 7 MP

QED

14.
1. A ∨ (B • C)
2. (C ∨ A) ⊃ ~~B
3. (A ∨ B) • (A ∨ C)   1, Dist
4. (A ∨ C) • (A ∨ B)   3, Com
5. A ∨ C   4, Simp
6. C ∨ A   5, Com
7. ~~B   2, 6, MT
8. B   7, DN

QED

15.
1. (K • L) • M
2. K ⊃ N
3. N ⊃ ~(O ∨ P)
4. K • (L • M)   1, assoc
5. K   4, simp
6. K ⊃ ~(O ∨ P)   2, 3 HS
7. ~(O ∨ P)   5, 6 MP
8. ~O • ~P   7, DM
9. ~P • ~O   8, com
10. ~P   9, simp

QED

16.
1. [O ∨ (P • Q)] ⊃ R
2. R ⊃ ~S
3. P • S
4. S • P   3, com
5. S   4, Simp
6. ~~S   5, DN
7. ~R   2, 6, MT
8. ~[O ∨ (P • Q)] 1, 7, MT
9. ~O • ~(P • Q)   8, DM
10. ~(P • Q) • ~O9, Com
11. ~(P • Q)   10, Simp
12. ~P ∨ ~Q   11, DM
13. P   3, Simp
14. ~~P   13, DN
15. ~Q   12, 14, DS

QED

17.
1. E ⊃ F
2. F ⊃ ~(G ∨ H)
3. I • E
4. E ⊃ ~(G ∨ H)   1, 2 HS
5. E • I   3, com
6. E   5, simp
7. ~(G ∨ H)   4, 6 MP
8. ~G • ~H   7, DM
9. ~H • ~G   8, com
10. ~H   9, simp

QED

18.
1. T ∨ (U • V)
2. T ⊃ (W • X)
3. ~W
4. (T ∨ U) • (T ∨ V)   1, dist
5. (T ∨ V) • (T ∨ U)   4, com
6. T ∨ V   5, simp
7. ~W ∨ ~X   3, add
8. ~(W • X)   7, DM
9. ~T   2, 8 MT
10. V   6, 9 DS

QED

19.
1. A ⊃ B
2. ~[(C • D) ∨ (C • B)]
3. C • E
4. ~(C • D) • ~(C • B)   2, DM
5. ~(C • B) • ~(C • D)   4, com
6. ~(C • B)   5, simp
7. ~C ∨ ~B   6, DM
8. C   3, simp
9. ~~C   8, DN
10. ~B   7, 9 DS
11. ~A   1, 10 MT

QED

20.
1. [T • (U ∨ V)] ⊃ W
2. W ⊃ ~X
3. Y • X

4. [T • (U ∨ V)] ⊃ ~X		1, 2 HS
5. X • Y		3, com
6. X		5, simp
7. ~~X		6, DN
8. ~[T • (U ∨ V)]		4, 7 MT
9. ~T ∨ ~(U ∨ V)		8, DM
10. ~T ∨ (~U • ~V)		9, DM
11. (~T ∨ ~U) • (~T ∨ ~V)		10, dist
12. ~(T • U) • ~(T • V)		11, DM
QED

21.
1. F ⊃ G
2. H ⊃ I
3. (J ∨ F) ∨ H
4. ~J • ~G
5. (F ⊃ G) • (H ⊃ I)		1, 2 conj
6. J ∨ (F ∨ G)		3, assoc
7. ~J		4, simp
8. F ∨ G		6, 7 DS
9. G ∨ I		5, 8 CD
10. ~G • ~J		4, com
11. ~G		10, simp
12. I		9, 11 DS
QED

22.
1. O ⊃ P
2. (O • ~Q) • ~R
3. P ⊃ [Q ∨ (R ∨ S]
4. O ⊃ [Q ∨ (R ∨ S]		1, 3 HS
5. O • (~Q • ~R)		2, assoc
6. O		5, simp
7. Q ∨ (R ∨ S)		4, 6 MP
8. (~Q • ~R) • O		5, com
9. ~Q • ~R		8, simp
10. ~(Q ∨ R)		9, DM
11. (Q ∨ R) ∨ S		7, assoc
12. S		10, 11 DS
QED

23.
1. U ⊃ V
2. V ⊃ ~(W • X)
3. U • (W • Y)
4. U ⊃ ~(W • X)		1, 2 HS
5. (U • W) • Y		3, assoc
6. U • W		5, simp
7. U		6, simp
8. ~(W • X)		4, 7 MP
9. ~W ∨ ~X		8, DM
10. W • U		6, com
11. W		10, simp
12. ~~W		11, DN
13. ~X		9, 12 DS
14. Y • (U • W)		5, Com
15. Y		14, Simp
16. ~X • Y		13, 15, Conj
QED

24.
1. K ⊃ ~L
2. K ∨ (M • N)
3. M ⊃ ~N
4. (K ∨ M) • (K ∨ N)		2, dist
5. (K ⊃ ~L) • (M ⊃ ~N)		1, 3 conj
6. K ∨ M		4, simp
7. ~L ∨ ~N		5, 6 CD
8. ~(L • N)		7, DM
QED

25.
1. (O • P) ⊃ (Q • R)
2. (P ⊃ ~Q) • (O ⊃ ~R)
3. P
4. P ∨ O		3, add
5. ~Q ∨ ~R		2, 4 CD
6. ~(Q • R)		5, DM
7. ~(O • P)		1, 6 MP
8. ~O ∨ ~P		7, DM
9. ~P ∨ ~O		8, com
QED

Exercises 2.3b

1.	1. A ⊃ B	
	2. (~B • C) • D	/ ~A
	3. ~B • (C • D)	2, assoc
	4. ~B	3, simp
	5. ~A	1, 4 MT
	QED	

2.	1. C ⊃ D	
	2. (C • E) ∨ (C • F)	/ D
	3. C • (E ∨ F)	2, dist
	4. C	3, simp
	5. D	1, 4 MP
	QED	

3.	1. M ⊃ N	
	2. L ∨ ~N	
	3. ~(L ∨ O)	/ ~M
	4. ~L • ~O	3, DM
	5. ~L	4, simp
	6. ~N	2, 5 DS
	7. ~M	1, 6 MT
	QED	

4.	1. ~[P ∨ (Q • R)]	
	2. (~Q ⊃ S) • (~R ⊃ T)	/ S ∨ T
	3. ~P • ~(Q • R)	1, DM
	4. ~P • (~Q ∨ ~R)	3, DM
	5. (~Q ∨ ~R) • ~P	4, com
	6. ~Q ∨ ~R	5, simp
	7. S ∨ T	2, 6 CD
	QED	

5.	1. U ∨ (V ∨ W)	
	2. ~U • ~V	
	3. T ⊃ ~W	/ ~T
	4. (U ∨ V) ∨ W	1, assoc
	5. ~(U ∨ V)	2, DM
	6. W	4, 5 DS
	7. ~ ~W	6, DN
	8. ~T	3, 7 MT
	QED	

6.	1. X ∨ (Y • Z)	
	2. X ⊃ Z	
	3. ~(W ∨ Z)	/ Y • Z
	4. ~W ∨ ~Z	3, DM
	5. ~Z • ~W	4, com
	6. ~Z	5, simp
	7. ~X	2, 6 MT
	8. Y • Z	1, 7 DS
	QED	

7.	1. A ⊃ B	
	2. B ⊃ [(C • D) ∨ E]	
	3. (A • ~C) ∨ (A • ~D)	/ E
	4. A ⊃ [(C • D) ∨ E]	1, 2 HS
	5. A • (~C ∨ ~D) 3, dist	
	6. A • ~(C • D)	5, DM
	7. ~(C • D) • A	6, com
	8. ~(C • D)	7, simp
	9. A	5, Simp
	10. (C • D) ∨ E	4, 9, MP
	11. E	10, 8, DS
	QED	

8.	1. ~G ⊃ H	
	2. H ⊃ I	
	3. ~(G ∨ J)	/ I • ~J
	4. ~G ⊃ I	1, 2 HS
	5. ~G • ~J	3, DM
	6. ~G	5, simp
	7. I	4, 6 MP
	8. ~J • ~G	5, com
	9. ~J	8, simp
	10. I • ~J	7, 9 conj
	QED	

9.	1. ~[(E • F) ∨ (G ∨ H)]	
	2. I ∨ J	
	3. I ⊃ H	/ J
	4. ~(E • F) • ~(G ∨ H)	1, DM
	5. (~E ∨ ~F) • (~G • ~H)	4, DM
	6. (~G • ~H) • (~E ∨ ~F)	5, com
	7. ~G • ~H	6, simp
	8. ~H • ~G	7, com
	9. ~H	8, simp
	10. ~I	3, 9 MT
	11. J	2, 10 DS
	QED	

10.	1. ~(K • L) ⊃ M	
	2. M ⊃ N	
	3. ~[(O • P) ∨ N]	/ L
	4. ~(O • P) • ~N	3, DM
	5. ~N • ~(O • P)	4, com
	6. ~N	5, simp
	7. ~(K • L) ⊃ N	1, 2 HS
	8. ~ ~(K • L)	6, 7 MT
	9. K • L	8, DN
	10. L • K	9, com
	11. L	10, simp
	QED	

Exercises 2.4a

1.	1. A ⊃ B	
	2. B ⊃ ~B	
	3. ~B ∨ ~B	2, impl

	4. ~B	3, taut
	5. ~A	1, 4 MT
	QED	
2.	1. ~K ∨ L	
	2. L ⊃ ~K	
	3. K ⊃ L	1, impl
	4. K ⊃ ~K	3, 2, HS
	5. ~K ∨ ~K	4, Impl
	6. ~K	5, Taut
	QED	
3.	1. (A ⊃ B) ⊃ C	
	2. ~A ∨ (B • D)	
	3. (~A ∨ B) • (~A ∨ D)	2, dist
	4. ~A ∨ B	3, simp
	5. A ⊃ B	4, impl
	6. C	1, 5 MP
	QED	
4.	1. G ⊃ H	
	2. ~(I ⊃ H)	
	3. ~(~I ∨ H)	2, impl
	4. ~~I • ~H	3, DM
	5. ~H • ~~I	4, com
	6. ~H	5, simp
	7. ~G	1, 6 MT
	QED	
5.	1. ~I ∨ J	
	2. J ≡ K	
	3. (I • L) ∨ (I • M)	
	4. I ⊃ J	1, impl
	5. (J ⊃ K) • (K ⊃ J)	2, Equiv
	6. J ⊃ K	5, Simp
	7. I ⊃ K	4, 6, HS
	8. I • (L ∨ M)	3, Dist
	9. I	8, Simp
	10. K	7, 9, MP
	QED	
6.	1. (T • U) ⊃ V	
	2. ~(T ⊃ W)	
	3. ~(~T ∨ W)	2, Impl
	4. ~~T • ~W`	3, DM
	5. ~~T	4, Simp
	6. T	5, DN
	7. T ⊃ (U ⊃ V)	1, Exp
	8. U ⊃ V	7, 6, MP
	QED	
7.	1. W ⊃ (X • Y)	
	2. (W • ~X) ∨ Z	
	3. ~W ∨ (X • Y)	1, impl
	4. (~W ∨ X) • (~W ∨ Y)	3, dist
	5. ~W ∨ X	4, simp
	6. (~~W • ~X) ∨ Z	2, DN
	7. ~(~W ∨ X) ∨ Z	6, DM
	8. ~~(~W ∨ X)	5, DN
	9. Z	7, 8 DS
	QED	
8.	1. N ⊃ (O • P)	
	2. ~N ⊃ Q	
	3. ~N ∨ (O • P)	1, Impl
	4. (~N ∨ O) • (~N ∨ P)	3, Dist
	5. ~N ∨ O	4, Simp
	6. N ⊃ O	5, Impl
	7. ~O ⊃ ~N	6, Cont
	8. ~O ⊃ Q	7, 2, HS
	QED	
9.	1. E ≡ F	
	2. ~(G ∨ E)	
	3. (E ⊃ F) • (F ⊃ E)	1, equiv
	4. ~G • ~E	2, DM
	5. (F ⊃ E) • (E ⊃ F)	3, com
	6. F ⊃ E	5, simp
	7. ~E • ~G	4, com
	8. ~E	7, simp
	9. ~F	6, 8 MT
	QED	
10.	1. G ∨ H	
	2. ~I • (J • ~G)	
	3. (~I • J) • ~G	2, assoc
	4. ~G • (~I • J)	3, com
	5. ~G	4, simp
	6. H	1, 5 MP
	7. H ∨ ~I	6, add
	QED	
11.	1. A ∨ (B ∨ A)	
	2. ~(B ∨ C)	
	3. A ⊃ D	
	4. (B ∨ A) ∨ A	1, com
	5. B ∨ (A ∨ A)	4, assoc
	6. B ∨ A	5, taut
	7. ~B • ~C	2, DM
	8. ~B	7, simp
	9. A	6, 8 DS
	10. D	3, 9 MP
	QED	

12.
1. (H • I) ⊃ J
2. H • (I ∨ K)
3. (H • I) ∨ (H • K) 2, dist
4. ~J ⊃ ~(H • I) 1, Cont
5. ~~(H • I) ∨ (H • K) 3, DN
6. ~(H • I) ⊃ (H • K) 5, Impl
7. ~J ⊃ (H • K) 4, 6, HS
8. ~~J ∨ (H • K) 7, Impl
9. (~~J ∨ H) • (~~J ∨ K) 8, Dist
10. (~~J ∨ K) • (~~J ∨ H) 9, Com
11. ~~J ∨ K 10, Simp
12. ~J ⊃ K 11, Impl

QED

13.
1. L ⊃ ~(~M ∨ K)
2. M ⊃ (~K ⊃ N)
3. ~N
4. (M • ~K) ⊃ N 2, Exp
5. ~(M • ~K) 4, 3, MT
6. ~M ∨~~K 5, DM
7. ~M ∨ K 6, DN
8. ~~(~M ∨ K) 7, DN
9. ~L 1, 8, MT

QED

14.
1. Q ⊃ R
2. R ⊃ (S ⊃ T)
3. Q ⊃ (S ⊃ T) 1, 2 HS
4. (Q • S) ⊃ T 3, exp
5. ~(Q • S) ∨ T 4, impl
6. (~Q ∨ ~S) ∨ T 5, DM
7. (~ S ∨ ~Q) ∨ T 6, com
8. T ∨ (~S ∨ ~Q) 7, com
9. ~~T ∨ (~S ∨ ~Q) 8, DN
10. ~T ⊃ (~S ∨~Q) 9, impl
11. ~T ⊃ (S ⊃ ~Q) 10, impl

QED

15.
1. D ≡ E
2. (E ∨ F) ⊃ G
3. ~(G ∨ H)
4. ~G • ~H 3, DM
5. ~G 4, simp
6. ~(E ∨ F) 2, 5 MT
7. ~E • ~F 6, DM
8. (D ⊃ E) • (E ⊃ D) 1, equiv
9. D ⊃ E 8, simp
10. ~E 7, simp
11. ~D 9, 10 MT

QED

16.
1. D ∨ (E ∨ F)
2. F ⊃ (G • H)
3. ~G
4. (D ∨ E) ∨ F 1, assoc
5. ~~(D ∨ E) ∨ F 4, DN
6. ~(D ∨ E) ⊃ F 5, impl
7. ~ G ∨ ~H 3, add
8. ~(G • H) 7, DM
9. ~F 2, 8 MT
10. ~~(D ∨ E) 6, 9 MT
11. D ∨ E 10, DN

QED

17.
1. (X ⊃ Y) ⊃ Z
2. W ⊃ ~Z
3. ~Z ⊃ ~(X ⊃ Y) 1, cont
4. W ⊃ ~(X ⊃ Y) 2, 3 HS
5. ~W ∨ ~(X ⊃ Y) 4, impl
6. ~W ∨~(~X ∨ Y) 5, impl
7. ~W ∨ (~~X • ~Y) 6, DM
8. ~W ∨ (X • ~Y) 7, DN
9. (~W ∨ X) • (~W ∨ ~Y) 8, dist
10. (~W ∨ ~Y) • (~W ∨ X) 9, com
11. ~W ∨ ~Y 10, simp
12. ~(W • Y) 11, DM

QED

18.
1. R ⊃ T
2. T ⊃ S
3. (U • S) ⊃ R
4. ~(~U ∨ T)
5. ~~U • ~T 4, DM
6. U • ~T 5, DN
7. U 6, simp
8. R ⊃ S 1, 2 HS
9. U ⊃ (S ⊃ R) 3, exp
10. S ⊃ R 7, 9 MP
11. (R ⊃ S) • (S ⊃ R) 8, 10 conj
12. R ≡ S 11, equiv

QED

19.
1. (S ≡ T) • ~U
2. ~S ∨ (~T ∨ U)
3. S ≡ T 1, simp
4. (S • T) ∨ (~S • ~T) 3, equiv
5. (~S ∨ ~T) ∨ U 2, assoc
6. ~(S • T) ∨ U 5, DM
7. U ∨ ~(S • T) 6, com
8. ~U • (S ≡ T) 1, com
9. ~ U 8, simp
10. ~(S • T) 7, 9 HS
11. ~S • ~T 4, 10 HS
12. ~S 11, simp

QED

20.

1. [V ∨ (W ∨ X)] ⊃ Y	
2. Y ⊃ Z	
3. [V ∨ (W ∨ X)] ⊃ Z	1, 2 HS
4. [(V ∨ W) ∨ X] ⊃ Z	3, assoc
5. ~[(V ∨ W) ∨ X] ∨ Z	4, impl
6. [~(V ∨ W) • ~X] ∨ Z	5, DM
7. [(~V • ~W) • ~X] ∨ Z	6, DM
8. Z ∨ [(~V • ~W) • ~X]	7, com
9. [Z ∨(~V • ~W)] • (Z ∨ ~X)	8, dist
10. [(Z ∨ ~V) • (Z ∨ ~W)] • (Z ∨ ~X)	9, dist
11. (Z ∨ ~V) • (Z ∨ ~W)	10, simp
12. Z ∨ ~V	11, simp
QED	

21.

1. A ⊃ (B ⊃ C)	
2. ~C ∨ (D • E)	
3. ~(D ∨ F)	
4. (~C ∨ D) • (~C ∨ E)	2, dist
5. ~C ∨ D	4, simp
6. ~D • ~F	3, DM
7. ~D	6, simp
8. D ∨ ~C	5, com
9. ~C	7, 8 DS
10. (A • B) ⊃ C	1, exp
11. ~(A • B)	9, 10 MT
12. ~A ∨ ~B	11, DM
QED	

22.

1. F ⊃ (G ⊃ H)	
2. G • ~H	
3. J ⊃ F	
4. (F • G) ⊃ H	1, exp
5. ~(F • G) ∨ H	4, impl
6. (~F ∨ ~G) ∨ H	5, DM
7. ~F ∨ (~G ∨ H)	6, assoc
8. (~G ∨ H) ∨ ~F	7, com
9. ~~G • ~H	2, DN
10. ~(~G ∨ H)	9, DM
11. ~F	8, 10 DS
12. ~J	3, 11 MT
QED	

23.

1. N ⊃ O	
2. P ⊃ Q	
3. ~(Q ∨ O)	
4. ~Q • ~O	3, DM
5. ~Q	4, simp
6. ~P	2,5 MT
7. ~O • ~Q	4, com
8. ~O	7, simp
9. ~N	1, 8 MT
10. ~ P • ~N	6, 9 conj
11. (~P • ~N) ∨ (P • N)	10, add
12. (P • N) ∨ (~ P • ~N)	11, com
13. P ≡ N	12, equiv
QED	

24.

1. O ≡ P	
2. P ≡ Q	
3. (O ⊃ P) • (P ⊃ O)	1, equiv
4. (P ⊃ Q) • (Q ⊃ P)	2, equiv
5. O ⊃ P	3, simp
6. P ⊃ Q	4, simp
7. O ⊃ Q	5, 6 HS
8. (P ⊃ O) • (O ⊃ P)	3, com
9. P ⊃ O	8, simp
10. (Q ⊃ P) • (P ⊃ Q)	4, com
11. Q ⊃ P	10, simp
12. Q ⊃ O	9, 11 HS
13. (O ⊃ Q) • (Q ⊃ O)	7, 12 conj
14. O ≡ Q	13, equiv
QED	

25.

1. T ⊃ (U ⊃ V)	
2. Q ⊃ (R ⊃ V)	
3. (T • U) ∨ (Q • R)	
4. (T • U) ⊃ V	1. Exp
5. (Q • R) ⊃ V	2, Exp
6. [(T • U) ⊃ V] • [(Q • R) ⊃ V]	4, 5, Conj
7. V ∨ V	6, 3, CD
8. V	7, Taut
QED	

26.

1. ~(X ⊃ Y)	
2. Y ∨ (Z • A)	
3. ~(~X ∨ Y)	1, Impl
4. ~~X • ~Y	3, DM
5. ~Y • ~~X	4, Com
6. ~Y	5, Simp
7. Z • A	2, 6, DS
8. (Z • A) ∨ (~Z • ~A)	7, Add
9. Z ≡ A	8, Equiv.
QED	

Exercises 2.4b

1.
1. R ≡ S
2. ~S / ~R
3. (R ⊃ S) • (S ⊃ R) 1, equiv
4. R ⊃ S 3, simp
5. ~R 2, 4 MT

QED

2.
1. A ⊃ B
2. B ⊃ C / C ∨ ~A
3. A ⊃ C 1, 2 HS
4. ~C ⊃ ~A 3, cont
5. ~~C ∨ A 4, impl
6. C ∨ ~A 5, DN

QED

3.
1. Q ⊃ (R ⊃ S) / ~[Q • (R • ~S)]
2. (Q • R) ⊃ S 1, exp
3. ~(Q • R) ∨ S 2, impl
4. (~Q ∨ ~R) ∨ S 3, DM
5. ~Q ∨ (~R ∨ S) 4, assoc
6. ~Q ∨ (~R ∨ ~~S) 5, DN
7. ~Q ∨ ~(R • ~S) 6, DM
8. ~[Q • (R • ~S)] 7, DM

QED

4.
1. L ⊃ M
2. N ⊃ (O ⊃ L)
3. O • N / M
4. (N • O) ⊃ L 2, Exp
5. N • O 3, Com
6. L 4, 5, MP
7. M 1, 6, MP

QED

5.
1. C ≡ D
2. (D • E) • F / C
3. (C ⊃ D) • (D ⊃ C) 1, equiv
4. (D ⊃ C) • (C ⊃ D) 3, com
5. D ⊃ C 4, simp
6. D • (E • F) 2, assoc
7. D 6, simp
8. C 5, 7 MP

QED

6.
1. O ⊃ P
2. P ⊃ ~O
3. (O ∨ Q) ∨ R / Q ∨ R
4. O ⊃ ~O 1, 2 HS
5. ~O ∨ ~O 4, impl
6. ~O 5, taut
7. O ∨ (Q ∨ R) 3, assoc
8. Q ∨ R 6, 7 DS

QED

7.
1. ~(V ⊃ W)
2. W ∨ U / U
3. ~(~V ∨ W) 1, impl
4. ~~V • ~W 3, DM
5. V • ~W 4, DN
6. ~W • V 5, com
7. ~W 6, simp
8. U 2, 7 DS

QED

8.
1. J ≡ K / ~K ≡ ~J
2. (J ⊃ K) • (K ⊃ J) 1, equiv
3. J ⊃ K 2, simp
4. ~K ⊃ ~J 3, cont
5. (K ⊃ J) • (J ⊃ K) 2, com
6. K ⊃ J 5, simp
7. ~J ⊃ ~K 6, cont
8. (~K ⊃ ~J) • (~J ⊃ ~K) 4, 7 conj
9. ~K ≡ ~J 8, equiv

QED

9.
1. ~L ∨ M
2. (L • N) • O
3. ~M ∨ P / P
4. L ⊃ M 1, impl
5. L • (N • O) 2, assoc
6. L 5, simp
7. M 4, 6 MP
8. ~~M 7, DN
9. P 3, 8 DS

QED

10.
1. (X ⊃ Y) ⊃ Z
2. Z ⊃ W / W ∨ X
3. (X ⊃ Y) ⊃ W 1, 2 HS
4. ~(X ⊃ Y) ∨ W 3, impl
5. ~(~X ∨ Y) ∨ W 4, impl
6. (~~X • ~Y) ∨ W 5, DM
7. (X • ~Y) ∨ W 6, DN
8. W ∨ (X • ~Y) 7, com
9. (W ∨ X) • (W ∨ ~Y) 8, dist
10. W ∨ X 9, simp

QED

Exercises 2.5a

1.
1. (A ∨ C) ⊃ D
2. D ⊃ B
| 3. A — ACP
| 4. (A ∨ C) ⊃ B — 1, 2 HS
| 5. A ∨ C — 3, add
| 6. B — 4, 5 MP
7. A ⊃ B — 3-6 CP
QED

2.
1. X ⊃ Y
2. Y ⊃ Z
| 3. X — ACP
| 4. Y — 1, 3 MP
| 5. Z — 2, 4 MP
| 6. Y • Z — 4, 5, Conj
7. X ⊃ (Y • Z) — 3-6 CP
QED

3.
1. Q ⊃ (~R • S)
| 2. Q — ACP
| 3. ~R • S — 1, 2 MP
| 4. ~R — 3, simp
5. Q ⊃ ~R — 2-4 CP
6. ~ ~R ⊃ ~Q — 5, cont
7. R ⊃ ~Q — 6, DN
QED

4.
1. ~(P • Q) ⊃ [(~P • ~Q) • (~R • ~S)]
| 2. ~(P • Q) — ACP
| 3. (~P • ~Q) • (~R • ~S) — 1, 2 MP
| 4. ~P • ~Q — 3, simp
5. ~(P • Q) ⊃ (~P • ~Q) — 2-4 CP
6. ~ ~(P • Q) ∨ (~P • ~Q) — 5, impl
7. (P • Q) ∨ (~P • ~Q) — 6, DN
8. P ≡ Q — 7, equiv
QED

5.
1. A ⊃ [(D ∨ B) ⊃ C]
| 2. A — ACP
| | 3. D — ACP
| | 4. (D ∨ B) ⊃ C — 1, 2 MP
| | 5. D ∨ B — 3, add
| | 6. C — 4, 5 MP
| 7. D ⊃ C — 3-6 CP
8. A ⊃ (D ⊃ C) — 2-7 CP
QED

6.
1. Z ⊃ ~Y
| 2. X • Y — ACP
| | 3. Z — ACP
| | 4. ~Y — 2, 4 MP
| | 5. Y • X — 2, Com
| | 6. Y — 5, Simp
| | 7. Y ∨ W — 6, Add
| | 8. W — 7, 4 DS
| 9. Z ⊃ W — 3-8 CP
10. (X • Y) ⊃ (Z ⊃ W) — 2-9 CP
QED

7.
1. ~M ⊃ N
2. L ⊃ ~N
| 3. L — ACP
| 4. ~N — 2, 3 MP
| 5. ~ ~M — 1, 4 MT
| 6. M — 5, DN
7. L ⊃ M — 3-6 CP
8. ~L ∨ M — 7, impl
QED

8.
1. R ⊃ ~O
2. ~R ⊃ [S • (P ∨ Q)]
| 3. O — ACP
| 4. ~ ~O — 3, DN
| 5. ~R — 1, 4 MT
| 6. S • (P ∨ Q) — 2, 5 MP
| 7. (P ∨ Q) • S — 6, com
| 8. P ∨ Q — 7, simp
9. O ⊃ (P ∨ Q) — 3-8 CP
QED

9.
1. ~G ∨ (E • ~F)
| 2. E ⊃ F — ACP
| 3. ~E ∨ F — 2, Impl
| 4. ~ ~(~E ∨ F) — 3, DN
| 5. ~(~ ~E • ~F) — 4, DM
| 6. ~(E • ~F) — 5, DN
| 7. (E • ~F) ∨ ~G — 6, Com
| 8. ~G — 7, 6 DS
9. (E ⊃ F) ⊃ ~G — 2-8 CP
QED

10.

1. I ⊃ H	
2. ~I ⊃ J	
3. J ⊃ ~H	
│4. ~H	ACP
│5. ~I	1,4 MP
│7. J	2, 5 MP
8. ~H ⊃ J	4-7 CP
9. (J ⊃ ~H) • (~H ⊃ J)	3, 8 conj
10. J ≡ ~H	9, equiv
QED	

11.

1. ~(U ∨ V)	
2. W ⊃ X	
│3. U ∨ W	ACP
│4. ~U • ~V	1, DM
│5. ~U	4, simp
│6. W	3, 5 DS
│7. X	2, 6 MP
│8. X ∨ ~V	7, add
│9. ~V ∨ X	8, com
│10. V ⊃ X	9, impl
11. (U ∨ W) ⊃ (V ⊃ X)	3-10 CP
QED	

12.

1. ~M ∨ N	
2. P	
│3. M ∨ ~P	ACP
│4. ~P ∨ M	3, com
│5. P ⊃ M	4, impl
│6. M ⊃ N	1, impl
│7. P ⊃ N	5, 6 HS
│8. N	2, 7 MP
│9. N ∨ O	8, add
│10. O ∨ N	9, com
11. (M ∨ ~P) ⊃ (O ∨ N)	3-10 CP
QED	

13.

1. ~(I ∨ ~K)	
2. L ⊃ J	
│3. I ∨ L	ACP
│4. ~I • ~~K	1, DM
│5. ~I • K	4, DN
│6. ~I	5, simp
│7. L	3, 6 DS
│8. J	2, MP
│9. K • ~I	5, com
│10. K	9, simp
│11. K • J	8, 10 conj
12. (I ∨ L) ⊃ (K • J)	3-11 CP
QED	

14. 1. X ⊃ [(T ∨ W) ⊃ S]
2. (W ⊃ S) ⊃ (Y ⊃ R)
3. ~Z ⊃ ~R

| 4. X — ACP
| 5. (T ∨ W) ⊃ S — 1, 4, MP
| | 6. W — ACP
| | 7. W ∨ T — 6, Add
| | 8. T ∨ W — 7, Com
| | 9. S — 5, 8, MP
| 10. W ⊃ S — 6-9, CP
| 11. Y ⊃ R — 2, 10, CP
| 12. R ⊃ Z — 3, Cont
| 13. Y ⊃ Z — 11, 12, HS

14 X ⊃ (Y ⊃ Z) — 4-13, CP

QED

15. 1. E ⊃ (F ⊃ G)
2. ~(H ∨ ~E)
3. G ⊃ H

| 4. F — ACP
| 5. (E • F) ⊃G — 1, exp
| 6. ~H • ~~E — 2, DM
| 7. ~~E • ~H — 6, com
| 8. E • ~H — 7, DN
| 9. E — 8, simp
| 10. E • F — 4, 10 conj
| 11. G — 5, 10 MP
| 12. H — 3, 11 MP

13. F ⊃ H — 4-12 CP

QED

16. 1. ~M ⊃ ~(~P ∨ Q)
2. ~(O ∨ N)

| 3. M ⊃ O — ACP
| 4. ~O • ~N — 2, DM
| 5. ~O — 4, simp
| 6. ~M — 3, 5 MT
| 7. ~(~P ∨ Q) — 1, 6 MP
| 8. ~~P • ~Q — 7, DM
| 9. P • ~Q — 8, DN
| 10. P — 9, simp
| 11. ~N • ~O — 4, com
| 12. ~N — 11, simp
| 13. P • ~N — 10, 12 conj

14. (M ⊃ O) ⊃ (P • ~N) — 3-13 CP

QED

17.

1. (T ⊃ ~Q) • ~W		
2. ~Q ⊃ [(W ∨ S) • (W ∨ T)]		
3. ~T ∨ (S ⊃ X)		
4. ~W • (T ⊃ ~Q)	1, Com	
5. ~W		4, Simp
	│6. ~Q	ACP
	│7. (W ∨ S) • (W ∨ T)	2, 4, MP
	│8. W ∨ (S • T)	5, Dist
	│9. S • T	8, 5, DS
	│10. T • S	9, Com
	│11. T	10, Simp
	│12. ~~T	11, DN
	│13. S ⊃ X	3, 12, DS
	│14. S	9, Simp
	│15. X	13, 14, MP
16. ~Q ⊃ X		6-15, CP
17. T ⊃ ~Q		1, Simp
18. T ⊃ X		17, 16, HS

QED

18.

1. [P ⊃ (Q ⊃ P)] ⊃ S		
2. [(P ⊃ Q) ⊃ (~Q ⊃ ~P)] ⊃ (S ⊃ T)		
	│3. P	ACP
	│4. P ∨ ~Q	3, Add
	│5. ~Q ∨ P	4, Com
	│6. Q ⊃ P	5, Impl
7. P ⊃ (Q ⊃ P)		3-6, CP
8. S		1, 7, MP
	│9. P ⊃ Q	ACP
	│10. ~Q ⊃ ~P	9, Cont
11. (P ⊃ Q) ⊃ (~Q ⊃ ~P)		9-10, CP
12. S ⊃ T		2, 11, MP
13. T		12, 8, MP

QED

19.

1. E ⊃ ~(F ⊃ G)		
2. F ⊃ (E • H)		
	│3. E	ACP
	│4. ~(F ⊃ G)	1, 3 MP
	│5. ~(~F ∨ G)	4, impl
	│6. ~~F • ~G	5, DM
	│7. F • G	6, DN
	│8. F	7, simp
9. E ⊃ F		3-8 CP
	│10. F	ACP
	│11. E • H	2, 10 MP
	│12. E	11, simp
13. F ⊃ E		10-12 CP
14. (E ⊃ F) • (F ⊃ E)		9, 13 conj
15. E ≡ F		14, equiv

QED

```
20.     1. M ⊃ (~K ∨ N)
        2. N ⊃ L
        3. M ∨ (K • ~L)
            | 4. M                          ACP
            | 5. ~K ∨ N                     1, 4 MP
            |       | 6. K                  ACP
            |       | 7. ~~K 6, DN
            |       | 8. N                  5, 7 DS
            |       | 9. L                  2, 8 MP
            | 10. K ⊃ L                     6-9 CP
        11. M ⊃ (K ⊃ L)                     4-10 CP
            | 12. K ⊃ L                     ACP
            | 13. ~K ∨ L                    12, impl
            | 14. M ∨ (~~K • ~L)            3, DN
            | 15. M ∨ ~(~K ∨ L)             14, DM
            | 16. ~(~K ∨ L) ∨ M             15, com
            | 17. ~~(~K ∨ L) 13, DN
            | 18. M                         16, 17 DS
        19. (K ⊃ L) ⊃ M                     12-18 CP
        20. [M ⊃ (K ⊃ L)] • [(K ⊃ L) ⊃ M]   11, 19 conj
        21. M ≡ (K ⊃ L)                     20, equiv
            QED
```

Exercises 2.5b

```
1.      1. ~R ⊃ T               / ~R ⊃ (T ∨ S)
            | 2. ~R             ACP
            | 3. T              1, 2 MP
            | 4. T ∨ S          3, add
        5. ~R ⊃ (T ∨ S)   2-4 CP
            QED
```

```
2.      1. ~(P ∨ R)             / O ⊃ (P ⊃ Q)
            | 2. O              ACP
            | 3. ~P • ~R        1, DM
            | 4. ~P             3, simp
            | 5. ~P ∨ Q         4, add
            | 6. P ⊃ Q          5, impl
        7. O ⊃ (P ⊃ Q)          2-5 CP
            QED
```

```
3.      1. W ⊃ (Y ⊃ X)
        2. ~Y ⊃ ~W              / W ⊃ X
            | 3. W              ACP
            | 4. Y ⊃ X          1, 3, MP
            | 5. W ⊃ Y          2, Cont
            | 6. W ⊃ X          5, 4, HS
            | 7. X              6, 3, MP
        8. W ⊃ X                3-7 CP
            QED
```

4.

1. (A • B) ⊃ C		
2. (C ∨ D) ⊃ E		/ A ⊃ (B ⊃ E)
	| 3. A • B	ACP
	| 4. C	1, 3 MP
	| 5. C ∨ D	4, add
	| 6. E	2, 5 MP
7. (A • B) ⊃ E		3-6 CP
8. A ⊃ (B ⊃ E)		7, exp
	QED	

5.

1. ~F ⊃ G		/ (E • ~G) ⊃ (F ∨ H)
	| 2. E • ~G	ACP
	| 3. ~G • E	2, com
	| 4. ~G	3, simp
	| 5. ~ ~F	1, 4 MT
	| 6. F	5, DN
	| 7. F ∨ H	6, add
8. (E • ~G) ⊃ (F ∨ G)		2-7 CP
	QED	

6.

1. K ⊃ I		
2. ~K ⊃ J		/ I ∨ J
	| 3. ~I	ACP
	| 4. ~K	1, 3 MT
	| 5. J	2, 4 MP
6. ~I ⊃ J	3-5 CP	
7. ~ ~I ⊃ J		6, DN
8. I ∨ J		7, impl
	QED	

7.

1. A ⊃ C		
2. B ⊃ D	/ (A • B) ⊃ (C • D)	
	| 3. A • B	ACP
	| 4. A	3, simp
	| 5. C	1, 4 MP
	| 6. B • A	3, com
	| 7. B	6, simp
	| 8. D	2, 7 MP
	| 9. C • D	5, 8 conj
10. (A • B) ⊃ (C • D)		3-9 CP
	QED	

8.

1. ~T ⊃ ~(S • R)		/ R ⊃ (S ⊃ T)	
	| 3. R		ACP
	|	| 4. S	ACP
	|	| 5. S • R	3, 4 conj
	|	| 6. ~ ~(S • R)	5, DN
	|	| 7. ~ ~T 1, 6 MT	
	|	| 8. T	7, DN
	| 9. S ⊃ T		4-8 CP
10. R ⊃ (S ⊃ T)			3-9 CP
	QED		

9.

	1. A ⊃ C	
	2. C ⊃ D	
	3. ~C ∨ B	/ A ⊃ (B • D)
	| 4. A	ACP
	| 5. A ⊃ D	1, 2 HS
	| 6. D	4, 5 MP
	| 7. C	1, 4 MP
	| 8. ~ ~C 7, DN	
	| 9. B	3, 8 DS
	| 10. B • D	6, 9 conj
	11. A ⊃ (B • D)	4-10 CP
	QED	

10.

	1. J ⊃ ~L	
	2. L ∨ K	
	3. (K ∨ M) ⊃ J	/ J ≡ K
	| 4. J	ACP
	| 5. ~L	1, 4 MP
	| 6. K	2, 5 DS
	7. J ⊃ K	4-6 CP
	| 8. K	ACP
	| 9. K ∨ M	8, add
	| 10. J	3, 9 MP
	11. K ⊃ J	8-10 CP
	12. (J ⊃ K) • (K ⊃ J)	7, 11 conj
	13. J ≡ K	12, equiv
	QED	

Exercises 2.6a

1.

			/ [A ∨ (B • C)] ⊃ (A ∨ C)
	| 1. A ∨ (B • C)	ACP	
	| 2. (A ∨ B) • (A ∨ C)		1, dist
	| 3. (A ∨ C) • (A ∨ B)		2, com
	| 4. A ∨ C		3, simp
	5. [A ∨ (B • C)] ⊃ (A ∨ C)	1-4 CP	
	QED		

2.

			/ [(A ⊃ B) • C] ⊃ (~B ⊃ ~A)
	| 1. (A ⊃ B) • C	ACP	
	| | 2. ~B		ACP
	| | 3. A ⊃ B		1, simp
	| | 4. ~A		2, 3 MT
	| 5. ~B ⊃ ~A		2-4 CP
	6. [(A ⊃ B) • C] ⊃ (~B ⊃ ~A)		1-5 CP
	QED		

3. $/ \ (O \vee P) \supset [\sim(P \vee Q) \supset O]$

$\mid$ 1. $O \vee P$	ACP
$\mid \quad \mid$ 2. $\sim(P \vee Q)$	ACP
$\mid \quad \mid$ 3. $\sim P \bullet \sim Q$	2, DM
$\mid \quad \mid$ 4. $\sim P$	3, simp
$\mid \quad \mid$ 5. $P \vee O$	1, com
$\mid \quad \mid$ 6. O	4, 5 DS
$\mid$ 7. $\sim(P \vee Q) \supset O$	2-6 CP
8. $(O \vee P) \supset [\sim(P \vee Q) \supset O]$	1-7 CP
QED	

4. $/ \ [V \bullet (W \vee X)] \supset (\sim X \supset W)$

$\mid$ 1. $V \bullet (W \vee X)$	ACP
$\mid \quad \mid$ 2. $\sim X$	ACP
$\mid \quad \mid$ 3. $(W \vee X) \bullet V$	1, com
$\mid \quad \mid$ 4. $W \vee X$	3, simp
$\mid \quad \mid$ 5. $X \vee W$	4, com
$\mid \quad \mid$ 6. W	2, 5 DS
$\mid$ 7. $\sim X \supset W$	2-6 CP
8. $[V \bullet (W \vee X)] \supset (\sim X \supset W)$	1-7 CP
QED	

5. $/ \ [(D \supset \sim E) \bullet (F \supset E)] \supset [D \supset (\sim F \vee G)]$

$\mid$ 1. $(D \supset \sim E) \bullet (F \supset E)$	ACP
$\mid \quad \mid$ 2. D	ACP
$\mid \quad \mid$ 3. $D \supset \sim E$	1, simp
$\mid \quad \mid$ 4. $\sim E$	2, 3 MP
$\mid \quad \mid$ 5. $(F \supset E) \bullet (D \supset \sim E)$	1, com
$\mid \quad \mid$ 6. $F \supset E$	5, simp
$\mid \quad \mid$ 7. $\sim F$	4, 6 MT
$\mid \quad \mid$ 8. $\sim F \vee G$	7, add
$\mid$ 9. $D \supset (\sim F \vee G)$	2-8 CP
10. $[(D \supset \sim E) \bullet (F \supset E)] \supset [D \supset (\sim F \vee G)]$	1-9 CP
QED	

6. $/ \ [(H \supset I) \supset \sim(I \vee \sim J)] \supset (\sim H \supset J)$

$\mid$ 1. $(H \supset I) \supset \sim(I \vee \sim J)$	ACP
$\mid \quad \mid$ 2. $\sim H$	ACP
$\mid \quad \mid$ 3. $\sim H \vee I$	2, add
$\mid \quad \mid$ 4. $H \supset I$	3, impl
$\mid \quad \mid$ 5. $\sim(I \vee \sim J)$	1, 4 MP
$\mid \quad \mid$ 6. $\sim I \bullet \sim\sim J$	5, DM
$\mid \quad \mid$ 7. $\sim I \bullet J$	6, DN
$\mid \quad \mid$ 8. $J \bullet \sim I$	7, com
$\mid \quad \mid$ 9. J	8, simp
$\mid$ 10. $\sim H \supset J$	2-9 CP
11. $[(H \supset I) \supset \sim(I \vee \sim J)] \supset (\sim H \supset J)$	1-10 CP
QED	

7.

	/ [(W ⊃ X) • (Y ∨ ~X)] ⊃ [~(Z ∨ Y) ⊃ ~W]
1. (W ⊃ X) • (Y ∨ ~X)	ACP
2. ~(Z ∨ Y)	ACP
3. ~Z • ~Y	2, DM
4. ~Y • ~Z	3, com
5. ~Y	4, simp
6. (Y ∨ ~X) • (W ⊃ X)	1, com
7. Y ∨ ~X	6, simp
8. ~X	5, 7 DS
9. W ⊃ X	1, simp
10. ~W	8, 9 MT
11. ~(Z ∨ Y) ⊃ ~W	2-10 CP
12. [(W ⊃ X) • (Y ∨ ~X)] ⊃ [~(Z ∨ Y) ⊃ ~W]	1-11 CP
QED	

8.

	/ [(R • S) ⊃ U] ⊃ {~U ⊃ [R ⊃ (S ⊃ T)]}
1. (R • S) ⊃ U	ACP
2. ~U	ACP
3. R	ACP
4. ~(R • S)	1, 2 MP
5. ~R ∨ ~S	4, DM
6. ~~R	3, DN
7. ~S	5, 6 DS
8. ~S ∨ T	7, add
9. S ⊃ T	8, impl
10. R ⊃ (S ⊃ T)	3-9 CP
11. ~U ⊃ [R ⊃ (S ⊃ T)]	2-11 CP
12. [(R • S) ⊃ U] ⊃ {~U ⊃ [R ⊃ (S ⊃ T)]}	1-12 CP
QED	

9.

	/ [(~K ⊃ N) • ~(N ∨ L)] ⊃ [(K ⊃ L) ⊃ M]
1. (~K ⊃ N) • ~(N ∨ L)	ACP
2. ~K ⊃ N	1, simp
3. ~(N ∨ L) • (~K ⊃ N)	1, com
4. ~(N ∨ L)	3, simp
5. ~N • ~L	4, DM
6. ~N	5, simp
7. ~~K	2, 6 MT
8. ~L • ~N	5, com
9. ~L	8, simp
10. ~~K • ~L	7, 9 conj
11. (~~K • ~L) ∨ M	10, add
12. ~(~K ∨ L) ∨ M	11, DM
13. (~K ∨ L) ⊃ M	12, impl
14. (K ⊃ L) ⊃ M	13, impl
15. [(~K ⊃ N) • ~(N ∨ L)] ⊃ [(K ⊃ L) ⊃ M]	1-14 CP
QED	

10. / [(D • E) ⊃ (F ∨ G)] ≡ [(~F • ~G) ⊃ (~D ∨ ~E)]

│ 1. (D • E) ⊃ (F ∨ G)	ACP
│ │ 2. ~F • ~G	ACP
│ │ 3. ~(F ∨ G)	2, DM
│ │ 4. ~(D • E)	1, 3 MT
│ │ 5. ~D ∨ ~E	4, DM
│ 6. (~F • ~G) ⊃ (~D ∨~E)	2-5 CP
7. [(D • E) ⊃ (F ∨ G)] ⊃ [(~F • ~G) ⊃ (~D ∨ ~E)]	1-6 CP
│ 8. (~F • ~G) ⊃ (~D ∨~E)	ACP
│ │ 9. D • E	ACP
│ │ 10. ~~D • ~~E	2, DN
│ │ 11. ~(~D ∨ ~E)	3, DM
│ │ 12. ~(~F • ~G)	1, 4 MT
│ │ 13. ~~F ∨ ~~G	12, DM
│ │ 14. F ∨ G	13, DN
│ 15. (D • E) ⊃ (F ∨ G)	9-14 CP
16. [(~F • ~G) ⊃ (~D ∨ ~E)] ⊃ [(D • E) ⊃ (F ∨ G)]	8-15 CP
17. {[(D • E) ⊃ (F ∨ G)] ⊃ [(~F • ~G) ⊃ (~D ∨ ~E)]} • {[(~F • ~G) ⊃ (~D ∨ ~E)] ⊃ [(D • E) ⊃ (F ∨ G)]}	7, 16 conj
18. [(D • E) ⊃ (F ∨ G)] ≡ [(~F • ~G) ⊃ (~D ∨ ~E)]	17, equiv

Exercises 2.6b*

1. (~A ⊃ B) ⊃ (~B ⊃ A) or [(~A ⊃ B) • ~B] ⊃ A
2. (~C ∨ D) ⊃ (C ⊃ D)
3. [E • (F ∨ G)] ⊃ (~E ⊃ G)
4. [~(H ∨ I) • (J ⊃ I)] ⊃ ~J
5. [K • (~L ∨ M)] ⊃ [(L ⊃ ~K) ⊃ M]
6. [N ⊃ (P • Q)] ⊃ [~(O ∨ P) ⊃ ~N]
7. [(R ⊃ S) • (S ⊃ T)] ⊃ [~(T ∨ U) ⊃ ~R]
8. [(V ⊃ W) • (~W ∨ X)] ⊃ {[V • (Y • Z)] ⊃ X}
9. { [A ∨ (B • C)] • (A ⊃ D)} ⊃ [~(D ∨ E) ⊃ C]
10. [(F ⊃ G) • (H ⊃ F)] ⊃ [(H • I) ⊃ (~G ⊃ I)]

*Some alternate formulations are possible.

Exercises 2.7a

```
1.      1. U ⊃ (V ∨ W)
        2. ~(W ∨ V)
              |3. U                          AIP
              |4. V ∨ W                      1, 3 MP
              |5. ~(V ∨ W)                   2, com
              |6.  (V ∨ W) • ~(V ∨ W)        4, 5 conj
        7. ~U                                3-6 IP
        QED

2.      1. Y ∨ ~Z
        2. Z • (~X ∨ W)
              |3. X                  ACP
              |      |4. ~Y          AIP
              |      |5. ~Z          1, 4 DS
              |      |6. Z           2, simp
              |      |7. Z • ~Z      6, 5, Conj
              |8. ~~Y          4-7 IP
              |9. Y                  8, DN
        10. X ⊃ Y                    3-9 CP
QED

3.      1. X ⊃ T
        2. Y ⊃ T
        3. T ⊃ Z
              |4. X ∨ Y              ACP
              |      |5. ~Z          AIP
              |      |6. ~T          3, 5, MT
              |      |7. ~Y          2, 6, MT
              |      |8. ~X          1, 6, MT
              |      |9. Y           4, 8, DS
              |      |10. Y • ~Y     9, 7, Conj
              |11. ~~Z               5-10, IP
              |12. Z                 11, DN
        13. (X ∨ Y) ⊃ Z              4-12, CP
QED

4.      1. A ⊃ B
        2. ~(C ∨ ~A)
              |3. ~B                 AIP
              |4. ~A                 1, 3 MT
              |5. ~C • ~~A           2, DM
              |6. ~C • A             5, DN
              |7. A • ~C             6, com
              |8. A                  7, simp
              |9. ~A • A             4, 8 conj
        10. ~~B                3-9 IP
        11. B                        10, DN
              QED
```

5.
1. S ⊃ T
2. S ∨ (~R • U)
| 3. R — ACP
| | 4. ~T — AIP
| | 5. ~S — 1, 4 MT
| | 6. (S ∨ ~R) • (S ∨ U) — 2, dist
| | 7. S ∨ ~R — 6, simp
| | 8. ~R — 5, 7 DS
| | 9. R • ~R — 3, 8 conj
| 10. ~~T — 4-9 IP
| 11. T — 10, DN
12. R ⊃ T — 3-11 CP
QED

6.
1. F ⊃ (E ∨ D)
2. ~E • (~D ∨ ~F)
| 3. F — AIP
| 4. E ∨ D — 1, 3 MP
| 5. (~E • ~D) ∨ (~E • ~F) — 2, dist
| 6. ~(E ∨ D) ∨ (~E • ~F) — 5, DM
| 7. ~~(E ∨ D) — 4, 6 DN
| 8. ~E • ~F — 6, 7 DS
| 9. ~F • ~E — 8, com
| 10. ~F — 9, simp
| 11. F • ~F — 3, 10 conj
12. ~F — 3-12 IP
QED

7.
1. ~(K • J)
2. I ∨ (L • J)
| 3. K — ACP
| | 4. ~I — AIP
| | 5. L • J — 2, 4 DS
| | 6. J • L — 5, com
| | 7. J — 6, simp
| | 8. ~K ∨ ~J — 1, DM
| | 9. ~~K — 3, DN
| | 10. ~J — 8, 9 DS
| | 11. J • ~J — 7, 10 conj
| 12. ~~I — 4-11 IP
| 13. I — 12, DN
14. K ⊃ I — 3-13 CP
15. ~K ∨ I — 14, impl
QED

8.

1. $X \supset (W \supset Z)$	
2. $Y \vee W$	
$\mid$ 3. $\sim Y$	ACP
$\mid$ 4. W	2, 3, DS
$\mid \quad \mid$ 5. X	ACP
$\mid \quad \mid$ 6. $W \supset Z$	1, 5, MP
$\mid \quad \mid$ 7. Z	6, 4, MP
$\mid$ 8. $X \supset Z$	5-7, CP
9. $\sim Y \supset (X \supset Z)$	3-8, CP
QED	

9.

1. $M \supset L$	
2. $\sim(K \bullet N) \supset (M \vee L)$	
$\mid$ 3. $\sim(K \vee L)$	AIP
$\mid$ 4. $\sim K \bullet \sim L$	3, DM
$\mid$ 5. $\sim L \bullet K$	4, com
$\mid$ 6. $\sim L$	5, simp
$\mid$ 7. $\sim M$	1, 6 MT
$\mid$ 8. $\sim K$	4, simp
$\mid$ 9. $\sim K \vee \sim N$	8, add
$\mid$ 10. $\sim(K \bullet N)$	9, DM
$\mid$ 11. $M \vee L$	1, 10 MP
$\mid$ 12. L	7, 11 DS
$\mid$ 13. $\sim L \bullet L$	6, 12 conj
14. $\sim\sim(K \vee L)$	1-13 IP
15. $K \vee L$	14, DN
QED	

10.

1. $A \equiv (B \bullet D)$	
2. $C \supset (E \vee F)$	
3. $(A \vee \sim E) \bullet (A \vee \sim F)$	
$\mid$ 4. C	ACP
$\mid$ 5. $E \vee F$	2, 4, MP
$\mid$ 6. $(\sim E \vee A) \bullet (A \vee \sim F)$	3, Com
$\mid$ 7. $(\sim E \vee A) \bullet (\sim F \vee A)$	6, Com
$\mid$ 8. $(E \supset A) \bullet (\sim F \vee A)$	7, Impl
$\mid$ 9. $(E \supset A) \bullet (F \supset A)$	8, Impl
$\mid$ 10. $A \vee A$	9, 5, CD
$\mid$ 11. A	10, Taut
$\mid$ 12. $B \bullet D$	1, 11, MP
$\mid$ 13. B	12, Simp
14. $C \supset B$	4-13, CP
QED	

11.

1. $H \supset G$	
2. $H \vee J$	
3. $\sim(J \vee \sim I)$	
| 4. $\sim(G \bullet I)$	AIP
| 5. $\sim G \vee \sim I$	4, DM
| 6. $G \supset \sim I$	5, impl
| 7. $H \supset \sim I$	1, 6 HS
| 8. $\sim J \bullet \sim\sim I$	3, DM
| 9. $\sim\sim I \bullet \sim J$	8, com
| 10. $\sim\sim I$	9, simp
| 11. $\sim H$	7, 10 MT
| 12. J	2, 11 DS
| 13. $\sim J$	8, simp
| 14. $J \bullet \sim J$	12, 13 conj
15. $\sim\sim(G \bullet I)$	4-14 IP
16. $G \bullet I$	15, DN
QED	

12.

1. $X \supset Y$	
2. $\sim(Z \supset W)$	
| 3. X	ACP
| | 4. $\sim(Y \bullet Z)$	AIP
| | 5. $\sim Y \vee \sim Z$	4, DM
| | 6. $Y \supset \sim Z$	5, impl
| | 7. $X \supset \sim Z$	1, 6 HS
| | 8. $\sim Z$	3, 7 MP
| | 9. $\sim(\sim Z \vee W)$	2, impl
| | 10. $\sim\sim Z \bullet \sim W$	9, DM
| | 11. $Z \bullet \sim W$	10, DN
| | 12. Z	11, simp
| | 13. $\sim Z \bullet Z$	8, 12 conj
| 14. $\sim\sim(Y \bullet Z)$	4-13 IP
| 15. $Y \bullet Z$	14, DN
16. $X \supset (Y \bullet Z)$	3-15 CP
QED	

13.

1. $M \supset (L \bullet \sim P)$	
2. $K \supset \sim(O \bullet \sim P)$	
3. $N \supset O$	
| 4. $K \bullet M$	ACP
| 5. K	4, Simp
| 6. $\sim(O \bullet \sim P)$	2, 5, MP
| 7. $\sim O \vee \sim\sim P$	6, DM
| 8. $M \bullet K$	4, Com
| 9. M	8, Simp
| 10. $L \bullet \sim P$	1, 9, MP
| 11. $\sim P \bullet L$	10, Com
| 12. $\sim P$	11, Simp
| 13. $\sim\sim\sim P$	12, DN
| 14. $\sim\sim P \vee \sim O$	7, Com
| 15. $\sim O$	14, 13, DS
| 16. $\sim N$	3, 15, MT
17. $(K \bullet M) \supset \sim N$	4-16 CP
QED	

14.

1. A ⊃ B	
2. ~C ⊃ ~(A ∨ ~D)	
3. ~D ∨ (B • C)	
| 4. A	ACP
| | 5. ~(B • C)	AIP
| | 6. ~B ∨ ~C	5, DM
| | 7. B ⊃ ~C	6, impl
| | 8. B ⊃ ~(A ∨ ~D)	2, 7 HS
| | 9. A ⊃ ~(A ∨ ~D)	1, 8 HS
| | 10. ~(A ∨ ~D)	4, 9 MP
| | 11. ~A • ~~D	10, DM
| | 12. ~~D • ~A	11, com
| | 13. ~~D	12, simp
| | 14. B • C	3, 13 DS
| | 15. ~(B • C) • (B • C)	5, 14 conj
| 16. ~~(B • C)	5,-15 IP
| 17. B • C	16, DN
18. A ⊃ (B • C)	4-17 CP
QED	

15.

1. P ≡ (Q ∨ ~R)	
2. T • ~(Q • P)	
| 3. P • R	AIP
| 4. [P ⊃ (Q ∨ ~R)] • [(Q ∨ ~R) ⊃ P]	1, equiv
| 5. P ⊃ (Q ∨ ~R)	4, simp
| 6. P	3, simp
| 7. Q ∨ ~R	5, 6 MP
| 8. ~R ∨ Q	7, com
| 9. R • P	3, com
| 10. R	9, simp
| 11. ~~R	10, DN
| 12. Q	8, 11 DS
| 13. T • (~Q ∨ ~P)	2, DM
| 14. (~Q ∨ ~P) • T	13, com
| 15. ~Q ∨ ~P	14, simp
| 16. ~~Q	12, DN
| 17. ~P	15, 16 DS
| 18. P • ~P	6, 17 conj
19. ~(P • R)	3-18 IP
QED	

16.

1. A ≡ ~(B ∨ C)	
2. (D ∨ E) ⊃ ~C	
3. ~(A • D)	
	4. D	
		5. ~B	AIP
		6. [A ⊃ ~(B ∨ C)] • [~(B ∨ C) ⊃ A]	1, equiv
		7. [~(B ∨ C) ⊃ A] • [A ⊃ ~(B ∨ C)]	6, com
		8. ~(B ∨ C) ⊃ A	7, simp
		9. D ∨ E	4, add
		10. ~C	2, 9 MP
		11. ~B • ~C	5, 10 conj
		12. ~(B ∨ C)	11, DM
		13. A	8, 12 MP
		14. ~A ∨ ~D	3, DM
		15. ~~A	13, DN
		16. ~D	14, 15 DS
		17. D • ~D	4, 16 conj
	18. ~~B	5-17 IP
	19. B	18, DM
20. D ⊃ B	4-19 CP
QED	

17.

1. U ⊃ (P • ~Q)	
2. T ⊃ (S ∨ U)	
3. ~T ⊃ ~R	
	4. P ⊃ Q	ACP
		5. R	ACP
			6. ~S	AIP
			7. U ⊃ (~~P • ~Q)	1, DN
			8. U ⊃ ~(~P ∨ Q)	7, DM
			9. U ⊃ ~(P ⊃ Q)	8, impl
			10. ~~(P ⊃ Q)	4, DN
			11. ~U	9, 10 MT
			12. ~S • ~U	6, 11 conj
			13. ~ (S ∨ U)	12, DM
			14. ~T	2, 13 MT
			15. ~R	3, 14 MP
			16. R • ~R	5, 15 conj
		17. ~~S	6-16 IP
		18. S	17, DN
	19. R ⊃ S	5-18 CP
20. (P ⊃ Q) ⊃ (R ⊃ S)	4-19 CP
QED	

18.

1. B ⊃ C		
2. E ≡ ~(B ∨ A)		
3. D ⊃ ~E		
│4. D		ACP
│ │5. ~(A ∨ C)		AIP
│ │6. ~A • ~C		5, DM
│ │7. ~C • ~A		6, com
│ │8. ~C		7, simp
│ │9. ~B		1, 8 MT
│ │10. ~A	6, simp	
│ │11. ~B • ~A		9, 10 conj
│ │12. ~(B ∨ A)		11, DM
│ │13. [E ⊃ ~(B ∨ A)] • [~(B ∨ A) ⊃ E]		2, equiv
│ │14. [~(B ∨ A) ⊃ E] • [E ⊃ ~(B ∨ A)]		13, com
│ │15. ~(B ∨ A) ⊃ E		14, simp
│ │16. E		12, 15 MP
│ │17. ~E		3, 4 MP
│ │18. E • ~E		16, 17 conj
│19. ~~(A ∨ C)		5-18 IP
│20. A ∨ C		19, DN
21. D ⊃ (A ∨ C)		4-20 CP
QED		

19.

1. Z ⊃ Y		
2. Z ∨ W		
3. Y ⊃ ~W		
4. W ≡ ~X		
5. (W ⊃ ~X) • (~X ⊃ W)		4, equiv
│6. X		ACP
│ │7. ~Y		AIP
│ │8. ~Z		1, 7 MT
│ │9. W		2, 8 DS
│ │10. W ⊃ ~X		5, simp
│ │11. ~X	9, 10 MP	
│ │12. X • ~X		6, 11 conj
│13. ~~Y		7-12 IP
│14. Y		13, DN
15. X ⊃ Y		6-14 CP
│16. Y		ACP
│17. ~W		3, 16 MP
│18. (~X ⊃ W) • (W ⊃ ~X)		5, com
│19. ~X ⊃ W		18, simp
│20. ~~X		17, 19 MT
│21. X		20, DN
22. Y ⊃ X		16-21 CP
23. (X ⊃ Y) • (Y ⊃ X)		15, 22 conj
24. X ≡ Y		23, equiv
QED		

20.

1. F ⊃ (K ≡ M)	
2. ~F ⊃ [L ⊃ (F ≡ H)]	
3. ~(M ∨ ~L)	
4. ~H ⊃ ~(~K • L)	
5. ~M • ~~L	3, DM
│6. F	ACP
│7. K ≡ M	1, 5 MP
│8. (K ⊃ M) • (M ⊃ K)	7, equiv
│9. K ⊃ M	8, simp
│10. ~M	5, simp
│11. ~K	9, 10 MT
│ │12. ~H	AIP
│ │13. ~(~K • L)	4, 12 MP
│ │14. ~~K ∨ ~L	13, DM
│ │15. K ∨ ~L	14, DN
│ │16. ~L	11, 15 DS
│ │17. ~~L • ~M	5, com
│ │18. ~~L	17, simp
│ │19. ~L • ~~L	16, 18 conj
│20. ~~H	12-19 IP
│21. H	20, DN
22. F ⊃ H	6- 21 CP
│23. H	ACP
│ │24. ~F	AIP
│ │25. L ⊃ (F ≡ H)	2, 24 MP
│ │26. ~~L • ~M	5, com
│ │27. L • ~M	26, DN
│ │28. L	27, simp
│ │29. F ≡ H	25, 28 MP
│ │30. (F ⊃ H) • (H ⊃ F)	29, equiv
│ │31. (H ⊃ F) • (F ⊃ H)	30, com
│ │32. H ⊃ F	31, simp
│ │33. F	23, 32 MP
│ │34. ~F • F	24, 33 conj
│35. ~~F	24-34 IP
│36. F	35, DN
37. H ⊃ F	23-36 CP
38. (F ⊃ H) • (H ⊃ F)	22, 37 conj
39. F ≡ H	38, equiv

QED

Exercises 2.7b

1.

1. $Q \supset P$		
2. $\sim P \vee \sim Q$		/ $\sim Q$
| 3. Q		AIP
| 4. P		1, 3 MP
| 5. $\sim(P \bullet Q)$		2, DM
| 6. $P \bullet Q$		3, 4 conj
| 7. $\sim(P \bullet Q) \bullet (P \bullet Q)$		5, 6 conj
8. $\sim Q$		3-7 IP
QED		

2.

1. $(M \bullet \sim N) \vee O$		
2. $\sim M$		/ O
| 3. $\sim O$		AIP
| 4. $O \vee (M \bullet \sim N)$	1, com	
| 5. $M \bullet \sim N$		3, 4 DS
| 6. M		5, simp
| 7. $\sim M \bullet M$		2, 6 conj
8. $\sim\sim O$		3-7 IP
9. O		8, DN
QED		

3.

1. $G \supset H$		
2. $H \supset (I \bullet \sim G)$		/ $\sim G$
| 3. G		AIP
| 4. $G \supset (I \bullet \sim G)$		1, 2 HS
| 5. $I \bullet \sim G$		3, 4 MP
| 6. $\sim G \bullet I$		5, com
| 7. $\sim G$		6, simp
| 8. $G \bullet \sim G$		3, 7 conj
9. $\sim G$		3-8 IP
QED		

4.

1. $J \supset K$		
2. $J \vee (L \bullet J)$		/ K
| 3. $\sim K$		AIP
| 4. $\sim J$		1, 3 MT
| 5. $L \bullet J$	2, 4 DS	
| 6. $J \bullet L$	5, com	
| 7. J		6, simp
| 8. $\sim J \bullet J$		4, 7 conj
9. $\sim\sim K$		3-8 IP
10. K		9, DN
QED		

```
5.      1. (X ∨ Y) ⊃ V
        2. W ⊃ ~V                                  / W ⊃ ~X
           |3. W                                   ACP
           |        |4. X                          AIP
           |        |5. X ∨ Y                      4, add
           |        |6. V                          1, 5 MP
           |        |7. ~ ~V            6, DN
           |        |8. ~W                         2, 7 MT
           |        |9. W • ~W                     3, 8 conj
           |10. ~X                      4-9 IP
        11. W ⊃ ~X                                 3-10 CP
           QED
```

```
6.      1. O ⊃ M
        2. O ∨ N                        / M ∨ N
           |3. ~(M ∨ N)                            AIP
           |4. ~M • ~N                             3, DM
           |5. ~M                                  4, simp
           |6. ~O                                  1, 5 MT
           |7. N                                   2, 6 DS
           |8. ~N • ~M                             4, com
           |9. ~N                                  8, simp
           |10. N • ~N                             7, 9 conj
        11. ~ ~(M ∨ N)                             3-10 IP
        12. M ∨ N                                  11, DN
           QED
```

```
7.      1. G ⊃ E
        2. G ∨ H
        3. F ⊃ ~H                                  / E ∨ ~F
           |4. ~E                                  ACP
           |        |5. F                          AIP
           |        |6. ~G                         1, 4 MT
           |        |7. H                          2, 6 DS
           |        |8. ~ ~H            7, DN
           |        |9. ~F                         3, 8 MT
           |        |10. F • ~F                    5, 9 conj
           |11. ~F                                 5-10 IP
        12. ~E ⊃ ~F                                4-11 CP
        13. ~ ~E ∨ ~F                              12, impl
        14. E ∨ ~F                                 13, DN
           QED
```

8.	1. O ≡ (P ∨ Q)	
	2. ~(Q ⊃ O)	/ ~(N • P)
	| 3. N • P	AIP
	| 4. P • N	3, com
	| 5. P	4, simp
	| 6. P ∨ Q	5, add
	| 7. [O ⊃ (P ∨ Q)] • [(P ∨ Q) ⊃ O]	1, equiv
	| 8. [(P ∨ Q) ⊃ O] • [O ⊃ (P ∨ Q)]	7, com
	| 9. (P ∨ Q) ⊃ O	8, simp
	| 10. O	6, 9 MP
	| 11. ~(~Q ∨ O)	2, impl
	| 12. ~~Q • ~O	11, DM
	| 13. ~O • ~~Q	12, com
	| 14. ~O	13, simp
	| 15. O • ~O	10, conj
	16. ~(N • P)	3-15 IP
	QED	

9.	1. (I • ~G) ⊃ F	
	2. H ∨ J	
	3. J ⊃ I	/ ~(F ∨ G) ⊃ H
	| 4. ~(F ∨ G)	ACP
	| | 5. ~H	AIP
	| | 6. J	2, 5 DS
	| | 7. I	3, 6 MP
	| | 8. I ⊃ (~G ⊃ F)	1, exp
	| | 9. ~(~G ⊃ F) ⊃ ~I	8, cont
	| | 10. ~(~F ⊃ ~~G) ⊃ ~I	9, cont
	| | 11. ~(~~F ∨ ~~G) ⊃ ~I	10, impl
	| | 12. ~(F ∨ G) ⊃ ~I	11, DN
	| | 13. ~I	4, 12 MP
	| | 14. I • ~I	7, 13 conj
	| 15. ~~H	5-14 IP
	| 16. H	15, DM
	17. ~(F ∨ G) ⊃ H	4-16 CP
	QED	

10.

1. T ⊃ ~R	
2. ~(S ∨ V)	
3. T • (U ∨ ~R)	/ ~(R ∨ S)
│4. R ∨ S	AIP
│5. ~~R ∨ S	4, DN
│6. ~R ⊃ S	5, impl
│7. T ⊃ S	1, 6 HS
│8. ~S • ~V	2, DM
│9. ~S	8, simp
│10. (T • U) ∨ (T • ~R)	3, dist
│11. ~T	7, 9 MT
│12. ~T ∨ ~U	11, add
│13. ~(T • U)	12, DM
│14. T • ~R	10, 13
│15. ~R • T	14, com
│16. ~R	15, simp
│17. S	4, 16 DS
│18. ~S • S	9, 17 conj
19. ~(R ∨ S)	4-18 IP

QED

Chapter 3

Exercises 3.1a

1. Ta
2. Sb
3. Dd
4. Lg
5. Cs
6. Gc • Gg
7. Dh ∨ Di
8. ~Rj
9. ~Wk
10. Bl ∨ Bm
11. Pn • Po
12. Fp ⊃ ~Ip
13. Pr ≡ Nr
14. Is ⊃ Js
15. Ch ≡ Is

Exercises 3.1b

1. (∀x)(Cx ⊃ Dx)
2. (∀x)(Mx ⊃ Fx)
3. (∃x)(Tx • Gx)
4. (∃x)(Fx • ~Bx)
5. (∃x)(Cx • Rx)
6. (∀x)(Fx ⊃ Sx)
7. (∃x)(Px • Wx)
8. ~(∀x)(Bx ⊃ Yx)
9. (∃x)(Cx • ~Fx)
10. (∀x)(Mx ⊃ Lx)
11. (∃x)(Bx • ~Sx)
12. ~(∀x)(Wx ⊃ Ex)
13. (∃x)(Px • Sx)
14. (∃x)(Dx • Sx)
15. (∀x)(Hx ⊃ Mx)
16. (∃x)(Mx • ~Dx)
17. ~(∀x)(Sx ⊃ Px)
18. (∃x)(Sx • ~Hx)
19. (∀x)(Dx ⊃ ~Ax)
20. (∀x)(Lx ⊃ Cx)

Exercises 3.1c

1. $(\exists x)[(Px \bullet Fx) \bullet Sx]$
2. $(\exists x)[(Px \bullet Fx) \bullet {\sim}Sx]$
3. $(\forall x)[(Rx \bullet Fx) \supset Sx]$
4. $(\forall x)[(Ox \bullet Fx) \supset {\sim}Sx]$
5. $(\exists x)[Px \bullet (Ix \bullet {\sim}Fx)]$
6. $(\forall x)[(Px \bullet Fx) \supset Sx]$
7. $(\forall x)[(Px \bullet Fx) \supset {\sim}Cx]$
8. $(\forall x)(Cx \supset Wx) \bullet (\forall x)(Dx \supset Wx)$
9. $(\exists x)(Cx \bullet Ex) \bullet (\forall x)(Dx \supset Ex)$
10. ${\sim}(\forall x)[(Dx \vee Cx) \supset Lx]$
11. $(\exists x)[Rx \bullet Bx) \bullet Kx]$
12. $(\forall x)[(Rx \bullet Bx) \supset {\sim}Ux]$
13. ${\sim}(\forall x)[(Rx \bullet Bx) \supset Dx]$
14. $(\exists x)[(Ax \bullet Px) \bullet Ix]$
15. $(\exists x)[(Ax \bullet Px) \bullet Cx]$
16. $(\forall x)[Px \supset (Ax \bullet Lx)]$
17. $(\exists x)[(Ax \bullet Tx) \bullet {\sim}Sx]$
18. $(\forall x)[(Tx \bullet Ax) \supset Wx]$
19. $(\exists x)[Ax \bullet (Tx \equiv Dx)]$
20. $(\forall x)[Px \supset (Tx \bullet Ax)]$
21. $(\exists x)[(Nx \bullet Px) \bullet Ex]$
22. ${\sim}(\forall x)[(Nx \bullet Px) \supset Ox]$
23. $(\forall x)[(Nx \bullet Px) \supset Ox] \supset (\forall x)[(Nx \bullet Px) \supset {\sim}Ex]$
24. $(\forall x)(Lx \supset Sx)$
25. $(\forall x)[(Sx \bullet Lx) \supset Gx]$
26. $(\forall x)[(Gx \vee Cx) \supset Mx)]$
27. $(\forall x)(Gx \supset Hx) \bullet (\exists x)(Cx \bullet Hx)$
28. $(\forall x)[(Rx \bullet Px) \supset Sx]$
29. $(\forall x)[(Px \bullet Sx) \supset (Cx \vee Hx)]$
30. $(\exists x)[(Px \bullet Sx) \bullet (Cx \bullet {\sim}Ix)]$
31. $(\forall x)(Px \supset Wx) \vee (\exists x)(Px \bullet Cx)$
32. $(\forall x)(Sx \supset Wx)$
33. $(\forall x)[Px \supset (Wx \supset Sx)]$
34. ${\sim}(\exists x)(Px \bullet Sx) \vee (\exists x)(Px \bullet Rx)$
35. $[(\exists x)(Cx \bullet Sx) \bullet (\exists x)(Px \bullet Ix)] \supset (\exists x)(Px \bullet Tx)$
36. $(\forall x)[(Sx \bullet Hx) \supset Cx]$
37. $(\forall x)[(Cx \bullet Hx) \supset Sx]$
38. $(\forall x)[(Sx \bullet Hx) \supset Cx]$
39. $(\forall x)[(Ux \bullet Cx) \supset Px]$
40. $(\forall x)[(Cx \bullet Px) \supset Ux]$
41. $(\exists x)[(Ux \bullet Fx) \bullet {\sim}Px]$
42. ${\sim}(\forall x)[(Ex \bullet Sx) \supset Dx]$
43. $(\forall x)[(Sx \bullet {\sim}Dx) \supset {\sim}Ex]$
44. $(\forall x)(Sx \supset Hx) \vee (\exists x)(Sx \bullet Ix)$
45. $(\forall x)\{Hx \supset [Cx \equiv (Tx \bullet Gx)]\}$
46. $(\forall x)[(Bx \vee Wx) \supset (Hx \bullet Sx)]$
47. $(\exists x)(Px \bullet Ax) \supset (\exists x)(Cx \bullet Mx)$
48. $(\forall x)\{Sx \supset [Px \supset (Vx \bullet Fx)]\}$
49. $(\exists x)(Sx \bullet Dx) \bullet (\exists x)(Px \bullet Fx)$
50. $(\exists x)[(Rx \bullet Bx) \bullet Ex] \bullet (\forall x)[(Rx \bullet Wx) \supset Ex]$
51. $[(\exists x)(Tx \bullet Bx) \bullet (\exists x)(Sx \bullet Lx)] \supset (\exists x)(Sx \bullet {\sim}Gx)$
52. $(\forall x)[Jx \supset ({\sim}Wx \vee Lx)]$
53. $(\forall x)\{Cx \supset \{Px \equiv [(\forall y)(My \supset Gy) \bullet (\forall y)(Sy \supset {\sim}Wy)]\}\}$

Exercises 3.2

1. a) $Px \bullet Qx$, Px, Qx
b) Both ‘x’s
c) There are no unbound variables
d) Closed
e) $(\exists x)$

2. a) $(Px \bullet Qx) \supset {\sim}Ra$, $Px \bullet Qx$, Px, Qx, ${\sim}Ra$, Ra
b) Both ‘x’s
c) There are no unbound variables
d) Closed
e) $(\forall x)$

3. a) $(\forall x)$: $Px \bullet Qx$, Px, Qx
$(\exists x)$: $(Px \vee Qy) \vee Rx$, $Px \vee Qy$, Px, Qy, Rx
b) $(\forall x)$: The two ‘x’s in its scope
$(\exists x)$: The two ‘x’s in its scope
c) The ‘y’ in ‘Qy’ is unbound
d) Open
e) $\supset$

4. a) Py
b) There are no bound variables
c) The ‘y’ is unbound
d) Open
e) $(\exists x)$

5. a) Px
b) Only the ‘x’ in ‘Px’ is bound
c) The ‘x’ in ‘Qx’ unbound
d) Open
e) $\supset$

6. a) $Px \vee ({\sim}Qy \bullet Rx)$, Px, ${\sim}Qy \bullet Rx$, ${\sim}Qy$, Qy, Rx
b) Both ‘x’s are bound
c) The ‘y’ is unbound
d) Open
e) $\sim$

7. a) $Pa \supset Qb$, Pa, Qb
b) There are no bound variables
c) There are no unbound variables
d) Closed
e) $(\forall y)$

8. a) $Ry \bullet Qx$, Ry, Qx, Pa
b) The ‘x’ is bound
c) The ‘y’ is unbound
d) Open
e) $\bullet$

9. a) (∃x): Rx • ~Qx, Rx, ~Qx, Qx
(∀x): Px ⊃ Qa, Px, Qa
b) (∃x): The ‘x’s in ‘Rx’ and ‘Qx’
(∀x): The ‘x’ in ‘Px’
c) There are no unbound variables
d) Closed
e) ≡

10. There are no quantifiers
b) There are no bound variables
c) There are no unbound variables
d) Closed
e) ⊃

11. a) (∀x): Px ∨ Qx, Px, Qx
(∀y): ~Qy ⊃ ~Py, ~Qy, Qy, ~Py, Py
b) (∀x): The two ‘x’s
(∀y): The two ‘y’s
c) There are no unbound variables
d) Closed
e) ⊃

12. a) (∃x): [(Px ∨ Rx) • Qy] ⊃ (∀y)[(Rx ⊃ Qy) • Pb],
(Px ∨ Rx) • Qy, Px ∨ Rx, Px, Rx, Qy, (∀x)[(Rx ⊃ Qy) • Pb], (Rx ⊃ Qy) • Pb, Rx ⊃ Qy, Rx, Qy, Pb
(∀y): (Rx ⊃ Qy) • Pb, Rx ⊃ Qy, Rx, Qy, Pb
b) (∃x): All three ‘x’s are bound
(∀y): The ‘y’ in its scope is bound
c) The ‘y’ in the first ‘Qy’ is unbound
d) Open
e) (∃x)

13. a) (Px ≡ Rx) ⊃ Qa, Px ≡ Rx, Px, Rx, Qa
b) Both ‘x’s are bound
c) There are no unbound variables
d) Closed
e) ~

14. a) Qx ∨ Px, Qx, Px
b) No variables are bound
c) Both ‘x’s are unbound
d) Open
e) ~

15. a) (∀x): (Px • Qy) ⊃ (∃y)[(Ry ⊃ Sy) • Tx],
Px • Qy, Px, Qy, (∃y)[(Ry ⊃ Sy) • Tx],
(Ry ⊃ Sy) • Tx, Ry ⊃ Sy, Ry, Sy, Tx
(∃y): (Ry ⊃ Sy) • Tx, Ry ⊃ Sy, Ry, Sy, Tx
b) (∀x): Both ‘x’s are bound
(∃y): The ‘y’s in ‘Ry’ and ‘Sy’ are bound.
c) The ‘y’ in ‘Qy’ is unbound
d) Open
e) (∀x)

Exercises 3.3

1.

1. (∃y)(Ny • Oy)	
2. Na • Oa	1, EI
3. Na	2, simp
4. Nb • Ob	1, EI
5. Ob • Nb	4, com
6. Ob	5, simp
7. Na • Ob	3, 7 conj

QED

2.

1. (∀x)Hx ∨ Ja	
2. (∀x)[(~Jx • Ix) ∨ (~Jx • Kx)]	
3. Ja ∨ (∀x)Hx	1, com
4. (~Ja • Ia) ∨ (~Ja • Ka)	2, UI
5. ~Ja • (Ia ∨ Ka)	4, dist
6. ~Ja	5, simp
7. (∀x)Hx	3, 6 DS

QED

3.

1. (∃x)(Px • ~Qx)	
2. Pa • ~Qa	1, EI
3. ~Qa • Pa	2, com
4. ~Qa	3, simp
5. ~Qa ∨ Ra	4, add
6. Qa ⊃ Ra	5, impl
7. (∃x)(Qx ⊃ Rx)	6, EG

QED

4.

1. (∃x)(Tx • Ux) ⊃ (∀x)Vx	
2. (∃x)[(Wx • Tx) • Ux]	
3. (Wa • Ta) • Ua	2, EI
4. Wa • (Ta • Ua)	3, assoc
5. (Ta • Ua) • Wa	4, com
6. Ta • Ua	5, simp
7. (∃x)(Tx • Ux)	6, EG
8. (∀x)Vx	1, 7 MP

QED

5.

1. (∀x)(Ax ⊃ Bx)	
2. (∀x)(Cx ⊃ ~Bx)	
3. Aa	/ ~Ca
4. Aa ⊃ Ba	1, UI
5. Ba	4, 3, MP
6. Ca ⊃ ~Ba	2, UI
7. ~~Ba	5, DN
8. ~Ca	6, 7, MT

QED

6.

1. (∀x)(Ax ⊃ Bx)	
2. (∀x)(Cx ⊃ ~Bx)	
3. Ax ⊃ Bx	1, UI
4. Cx ⊃ ~Bx	2, UI
5. ~Bx ⊃ ~Ax	3, cont
6. Cx ⊃ ~Ax	4, 5 HS
7. (∀x)(Cx ⊃ ~Ax)	6, UG

QED

7.

1. (∀x)(Jx • Kx)	
2. Ja • Ka	1, UI
3. Ja	2, simp
4. Ka • Ja	2, com
5. Ka	4, simp
6. (∃x)Jx	3, EG
7. (∃x)Kx	5, EG
8. (∃x)Jx • (∃x)Kx	6, 7 conj

QED

8.

1. (∃x)(Dx • ~Ex)	
2. (∀x)(Ex ∨ Fx)	
3. Da • ~Ea	1, EI
4. Ea ∨ Fa	2, UI
5. ~Ea • Da	3, com
6. ~Ea	5, simp
7. Fa	4, 6 DS
8. (∃x)Fx	7, EG

QED

9.

1. (∃x)(Ax • ~Bx)	
2. (∀x)(Cx ⊃ Bx)	/ (∃x)(Ax • ~Cx)
3. Aa • ~Ba	1, EI
4. ~Ba • Aa	3, Com
5. ~Ba	4, Simp
6. Ca ⊃ Ba	2, UI
7. ~Ca	6, 5, MT
8. Aa	3, Simp
9. Aa • ~Ca	8, 7, Conj
10. (∃x)(Ax • ~Cx)	9, EG

QED

10.

1. (∃x)(Px • Qx)	
2. (∃x)(Rx • Sx)	
3. Pa • Qa	1, EI
4. Rb • Sb	2, EI
5. Pa	3, simp
6. Rb	4, simp
7. Pa • Rb	5, 6 conj

QED

11.

1. (∃x)(Fx • Hx) ≡ Gb	
2. Gb / Fa	
3. [(∃x)(Fx • Hx) ⊃ Gb)] • [Gb ⊃ (∃x)(Fx • Hx)]	1, equiv
4. [Gb ⊃ (∃x)(Fx • Hx)] • [(∃x)(Fx • Hx) ⊃ Gb)]	3, com
5. Gb ⊃ (∃x)(Fx • Hx)	4, simp
6. (∃x)(Fx • Hx)	5, 2, MP
7. Fa • Ha	6, EI
8. Fa	7, simp

QED

12.

1. (∀x)(Fx ≡ Gx)	
2. Fx ≡ Gx	1, UI
3. (Fx ⊃ Gx) • (Gx ⊃ Fx)	2, equiv
4. Fx ⊃ Gx	3, simp
5. (∀x)(Fx ⊃ Gx)	4, UG
6. (Gx ⊃ Fx) • (Fx ⊃ Gx)	3, com
7. Gx ⊃ Fx	6, simp
8. (∀x)(Gx ⊃ Fx)	7, UG
9. (∀x)(Fx ⊃ Gx) • (∀x)(Gx ⊃ Fx)	5, 8 conj

QED

13.

1. (∀x)Ax ⊃ Ba	
2. (∀x)~(Ax ⊃ Cx)	
3. ~(Ax ⊃ Cx)	2, UI
4. ~(~Ax ∨ Cx)	3, impl
5. ~~Ax • ~Cx	4, DM
6. ~~Ax	5, simp
7. Ax	6, DN
8. (∀x)Ax	7, UG
9. Ba	1, 8 MP
10. (∃x)Bx	9, EG
QED	

14.

1. (∃x)Lx ≡ Nb	
2. (∃x)[(Lx • Mx) • Ox]	
3. (La • Ma) • Oa	2, EI
4. La • (Ma • Oa)	3, assoc
5. La	4, simp
6. (∃x)Lx	5, EG
7. [(∃x)Lx ⊃ Nb] • [Nb ⊃ (∃x)Lx]	1, equiv
8. (∃x)Lx ⊃ Nb	7, simp
9. Nb	6, 8 MP
10. (∃x)Nx	9, EG
QED	

15.

1. (∀x)(Fx ∨ Hx) ⊃ (∃x)Ex	
2. (∀x)[Fx ∨ (Gx • Hx)]	
3. Fx ∨ (Gx • Hx)	2, UI
4. (Fx ∨ Gx) • (Fx ∨ Hx)	3, dist
5. (Fx ∨ Hx) • (Fx ∨ Gx)	4, com
6. Fx ∨ Hx	5, simp
7. (∀x)(Fx ∨ Hx)	6, UG
8. (∃x)Ex	1, 7 MP
9. Ea	8, EI
10. (∃y)Ey	9, EG
QED	

16.

1. (∀x)(Ix ⊃ Kx)	
2. (∀x)(Jx ⊃ Lx)	
3. (∃x)(Jx ∨ Ix)	
4. Ja ∨ Ia	3, EI
5. Ia ⊃ Ka	1, UI
6. Ja ⊃ La	2, UI
7. Ia ∨ Ja	4, com
8. (Ia ⊃ Ka) • (Ja ⊃ La)	5, 6 conj
9. Ka ∨ La	7, 8 CD
10. (∃x)(Kx ∨ Lx)	9, EG
QED	

17.

1. (x)[Gx ⊃ (Hx ∨ Ix)]	
2. (∃x)(Gx • ~Ix) / (∃x)(Gx • Hx)	
3. Ga • ~Ia	2, EI
4. Ga	3, Simp
5. Ga ⊃ (Ha ∨ Ia) 1, UI	
6. Ha ∨ Ia	5, 4, MP
7. ~Ia • Ga	3, Com
8. ~Ia	7, Simp
9. Ia ∨ Ha	6, Com
10. Ha	9, 8, DS
11. Ga • Ha	4, 10, Conj
12. (∃x)(Gx • Hx)	11, EG

QED

18.

1. (∀x)(Ax ≡ Cx)	
2. (∀x)(Bx ⊃ Cx)	
3. Ba	
4. Aa ≡ Ca	1, UI
5. Ba ⊃ Ca	2, UI
6. (Aa ⊃ Ca) • (Ca ⊃ Aa)	4, equiv
7. (Ca ⊃ Aa) • (Aa ⊃ Ca)	6, com
8. Ca ⊃ Aa	7, simp
9. Ba ⊃ Aa	5, 8 HS
10. Aa	3, 9 MP
11. (∃x)Ax	10, EG

QED

19.

1. (∃x)(Ax • Bx)	
2. (∀x)(Ax ⊃ Cx)	
3. (∀x)(Bx ⊃ Dx)	/ (∃x)(Cx • Dx)
4. Aa • Ba	1, EI
5. Aa	4, Simp
6. Aa ⊃ Ca	2, UI
7. Ca	6, 5, MP
8. Ba • Aa	4, Com
9. Ba	8, Simp
10. Ba ⊃ Da	3, UI
11. Da	10, 8, MP
12. Ca • Da	7, 11, Conj
13. (∃x)(Cx • Dx)	12, Conj

QED

20.

1. (∀x)(Mx ⊃ Nx)	
2. (∀x)(Ox ⊃ Px)	
3. (∀x)[Mx ∨ (Ox • Qx)]	
4. Mx ⊃ Nx	1, UI
5. Ox ⊃ Px	2, UI
6. Mx ∨ (Ox • Qx)	3, UI
7. (Mx ∨ Ox) • (Mx ∨ Qx) 6, dist	
8. Mx ∨ Ox	7, simp
9. (Mx ⊃ Nx) • (Ox ⊃ Px)	4,5 conj
10. Nx ∨ Px	8, 9 CD
11. (∀x)(Nx ∨ Px)	10, UG

QED

21.

1. (∃x)(Ax • Bx) ∨ (~Ca • Da)	
2. (∀x)(Dx ⊃ Cx)	/ (∃x)(Ax • Bx)
3. Da ⊃ Ca	2, UI
4. ~Da ∨ Ca	3, Impl
5. ~~(~Da ∨ Ca)	4, DN
6. ~(~~Da • ~Ca)	5, DM
7. ~(Da • ~Ca)	6, Com
8. ~(~Ca • Da)	7, Com
9. (~Ca • Da) ∨ (∃x)(Ax • Bx)	1, Com
10. (∃x)(Ax • Bx)	9, 8, DS

QED

22.

1. (∀x)(Fx • Gx)	
2. (∃x)(~Gx ∨ Ex)	
3. ~Ga ∨ Ea	2, EI
4. Fa • Ga	1, UI
5. Ga ⊃ Ea	3, impl
6. Ga • Fa	4, com
7. Ga	6, simp
8. Ea	5,7 MP
9. Fa	4, simp
10. Fa • Ea	9, 8 conj
11. (∃x)(Fx • Ex) 10, EG	

QED

23.

1. (∀x)(Mx ⊃ Nx)	
2. (∃x)(~Nx • Ox)	
3. (∃x)~Mx ⊃ (∃x)~Ox	/ (∃x)Ox • (∃x)~Ox
4. ~Na • Oa	2, EI
5. ~Na	4, simp
6. Ma ⊃ Na	1, UI
7. ~Ma	6, 5 MT
8. (∃x)~Mx	7, EG
9. (∃x)~Ox	3, 8, MP
10. Oa • ~Na	4, Com
11. Oa	10, Simp
12. (∃x)Ox	11, EG
13. (∃x)Ox • (∃x)~Ox	12, 9, Conj

QED

24.

1. (∀x)(Dx ⊃ Ex)	
2. (∀x)(Ex ⊃ ~Gx)	
3. (∃x)Gx	/ (∃x)~Dx
4. Ga	3, EI
5. ~~Ga 4, DN	
6. Ea ⊃ ~Ga	2, UI
7. ~Ea	6, 5, MT
8. Da ⊃ Ea	1, UI
9. ~Da	8, 7, MT
10. (∃x)~Dx	9, EG

QED

25.

1. (∃x)(Mx • Ox) ⊃ (∃x)Nx	

	2. (∃x)(Px • Mx)	
	3. (∀x)(~Px ∨ Ox)	
	4. Pa • Ma	2, EI
	5. ~Pa ∨ Oa	3, UI
	6. Pa	4, simp
	7. ~~Pa	6, DN
	8. Oa	5, 7 DS
	9. Ma • Pa	4, com
	10. Ma	9, simp
	11. Ma • Oa	8, 10 conj
	12. (∃x)(Mx • Ox)	11, EG
	13. (∃x)Nx	1, 12 MP
	QED	

26.	1. (∀x)(Dx • Ex)	
	2. (∃x)(~Fx ∨ Gx)	
	3. ~Fa ∨ Ga	2, EI
	4. Da • Ea	1, UI
	5. (Da • Ea) • (~Da • ~Ea)	4, add
	6. Da ≡ Ea	5, equiv
	7. Fa ⊃ Ga	3, impl
	8. (Da ≡ Ea) • (Fa ⊃ Ga)	6,7 conj
	9. (∃x)[(Dx ≡ Ex) • (Fx ⊃ Gx)]	8, EG
	QED	

27.	1. (∀x)[Tx ∨ (Ux • Vx)]	
	2. (∀x)(Wx ⊃ ~Tx)	
	3. Tx ∨ (Ux • Vx)	1, UI
	4. Wx ⊃ ~Tx	2, UI
	5. (Tx ∨ Ux) • (Tx ∨ Vx)	3, dist
	6. Tx ∨ Ux	5, simp
	7. ~~Tx ∨ Ux	6, DN
	8. ~Tx ⊃ Ux	7, impl
	9. Wx ⊃ Ux	4, 8 HS
	10. (∀x)(Wx ⊃ Ux)	9, UG
	QED	

28.	1. (∀x)(Lx ≡ Nx)	
	2. (∀x)(Nx ⊃ Mx)	
	3. (∀x)~(Mx ∨ Ox)	
	4. La ≡ Na	1, UI
	5. Na ⊃ Ma	2, UI
	6. ~(Ma ∨ Oa)	3, UI
	7. (La ⊃ Na) • (Na ⊃ La)	4, equiv
	8. La ⊃ Na	7, simp
	9. La ⊃ Ma	5, 8 HS
	10. ~Ma • ~Oa	6, DM
	11. ~Ma	10, simp
	12. ~La	9, 11 DM
	13. (∃x)~Lx	12, EG
	QED	

29.	1. (∃x)[Hx • (Ix ∨ Jx)]	
	2. (∀x)(Kx ⊃ ~Ix)	
	3. (∀x)(Hx ⊃ Kx)	
	4. Ha • (Ia ∨ Ja)	1, EI
	5. Ka ⊃ ~Ia	2, UI
	6. Ha ⊃ Ka	3, UI
	7. Ha ⊃ ~Ia	6, 5, HS
	8. (Ha • Ia) ∨ (Ha • Ja)	4, Dist
	9. ~Ha ∨ ~Ia	7, impl
	10. ~(Ha • Ia)	9, DM
	11. Ha • Ja	8, 10 DS
	12. Ja • Ha	11, com
	13. Ja	12, simp
	14. (∃x)Jx	13, EG
QED		

30.	1. (∃x)(Dx • Fx)	
	2. (∃x)(Gx ⊃ Ex)	
	3. (∀x)~(Hx ∨ Ex)	
	4. Da • Fa	1, EI
	5. Gb ⊃ Eb	2, EI
	6. ~(Hb ∨ Eb)	3, UI
	7. ~Hb • ~Eb	6, DM
	8. ~Eb • ~Hb	7, com
	9. ~Eb	8, simp
	10. ~Gb	5, 9 MT
	11. Fa • Da	4, com
	12. Fa	11, simp
	13. (∃x)Fx	12, EG
	14. (∃x)~Gx	10, EG
	15. (∃x)Fx • (∃x)~Gx	13, 14 conj
QED		

31.	1. (∀x)(Rx ≡ Tx)	
	2. (∃x)(Tx • ~Sx)	
	3. (∀x) [Sx ∨ (Rx ⊃ Ux)]	/ (∃x)Ux
	4. Ta • ~Sa	2, EI
	5. Ta	4, Simp
	6. Ra ≡ Ta	1, UI
	7. (Ra ⊃ Ta) • (Ta ⊃ Ra)	6, Equiv
	8. (Ta ⊃ Ra) • (Ra ⊃ Ta)	7, Com
	9. Ta ⊃ Ra	8, Simp
	10. Ra	9, MP
	11. ~Sa • Ta	4, Com
	12. ~Sa	11, Simp
`	13. Sa ∨ (Ra ⊃ Ua)	3, UI
	14. Ra ⊃ Ua	13, 12, DS
	15. Ua	14, 10, MP
	16. (∃x)Ux	15, EG
QED		

32.	1. $(\forall x)Ix \supset (\forall x)Kx$	
	2. $(\forall x)[Jx \bullet (Ix \vee Lx)]$	
	3. $(\forall x)(Jx \supset {\sim}Lx)$	
	4. $Jx \bullet (Ix \vee Lx)$	2, UI
	5. $Jx \supset {\sim}Lx$	3, UI
	6. Jx	4, simp
	7. ${\sim}Lx$	5,6 MP
	8. $(Ix \vee Lx) \bullet Jx$	4, com
	9. $Ix \vee Lx$	8, simp
	10. $Lx \vee Ix$	9, com
	11. Ix	7, 10 DS
	12. $(\forall x)Ix$	11, UG
	13. $(\forall x)Kx$	1, 12 MP
	QED	

33.	1. $(\forall x)(Px \vee Qx) \equiv Rc$	
	2. $(\forall x){\sim}(Sx \vee {\sim}Qx)$	
	3. ${\sim}(Sx \vee {\sim}Qx)$	2, UI
	4. ${\sim}Sx \bullet {\sim}{\sim}Qx$	3, DM
	5. ${\sim}Sx \bullet Qx$	4, DN
	6. $Qx \bullet {\sim}Sx$	5, com
	7. Qx	6, simp
	8. $Qx \vee Px$	7, add
	9. $Px \vee Qx$	8, com
	10. $(\forall x)(Px \vee Qx)$	9, UG
	11. $[(\forall x)(Px \vee Qx) \supset Rc] \bullet [Rc \supset (\forall x)(Px \vee Qx)]$	1, equiv
	12. $(\forall x)(Px \vee Qx) \supset Rc$	11, simp
	13. Rc	10, 12 MP
	14. $(\exists y)Ry$	13, EG
	QED	

34.	1. $(\exists x)Qx \equiv (\exists x)Sx$	
	2. $(\forall x)(Rx \vee Sx)$	
	3. $(\exists x){\sim}(Rx \vee Qx)$	
	4. ${\sim}(Ra \vee Qa)$	3, EI
	5. $Ra \vee Sa$	2, UI
	6. ${\sim}Ra \bullet {\sim}Qa$	4, DM
	7. ${\sim}Ra$	6, simp
	8. Sa	5, 7 DS
	9. $(\exists x)Sx$	8, EG
	10. $[(\exists x)Qx \supset (\exists x)Sx] \bullet [(\exists x)Sx \supset (\exists x)Qx]$	1, equiv
	11. $[(\exists x)Sx \supset (\exists x)Qx] \bullet [(\exists x)Qx \supset (\exists x)Sx]$	10, com
	12. $(\exists x)Sx \supset (\exists x)Qx$	11, simp
	13. $(\exists x)Qx$	9, 12 MP
	14. Qb	13, EI
	QED	

35.	1. (∃x)Ax ⊃ (∀x)Cx	
	2. (∀x)(~Bx ⊃ Dx)	
	3. (∀x)(Bx ⊃ Ax)	
	4. (∃x)~(Dx ∨ ~Cx)	
	5. ~(Da ∨ ~Ca)	4, EI
	6. ~Ba ⊃ Da	2, UI
	7. Ba ⊃ Aa	3, UI
	8. ~Da ⊃ ~~Ba	6, cont
	9. ~Da ⊃ Ba	8, DN
	10. ~Da • ~~Ca	5, DM
	11. ~Da	10, simp
	12. ~Da ⊃ Aa	7, 9 HS
	13. Aa	11, 12 MP
	14. (∃x)Ax	13, EG
	15. (∀x)Cx	11, 14 MP
	QED	

36.	1. (∀x)(Kx ⊃ Lx)	
	2. (∀x)(Lx ⊃ Mx)	
	3. Ka • Kb	
	4. Ka ⊃ La	1, UI
	5. Ka	3, simp
	6. La	4,5 MP
	7. (∃x)Lx	6, EG
	8. Kb ⊃ Lb	1, UI
	9. Lb ⊃ Mb	2, UI
	10. Kb ⊃ Mb	8, 9 HS
	11. Kb • Ka	3, com
	12. Kb	11, simp
	13. Mb	10, 12 MP
	14. (∃y)My	13, EG
	15. (∃x)Lx • (∃y)My	7, 14 conj
	QED	

37.	1. (∀x)(Ox ⊃ Qx)	
	2. (∀x)(Ox ∨ Px)	
	3. (∃x)(Nx • ~Qx)	/ (∃x)(Nx • Px)
	4. Na • ~Qa	3, EI
	5. Na	4, simp
	6. ~Qa • Na	4, Com
	7. ~Qa	6, Simp
	8. Oa ⊃ Qa	1, UI
	9. ~Oa	8, 7, MT
	10. Oa ∨ Pa	2, UI
	11. Pa	10, 9, DS
	12. Na • Pa	5, 11, Conj
	13. (∃x)(Nx • Px)12, EG	

QED

38.

1. $(\forall x)(Px \supset Qx)$	
2. $(\forall x){\sim}(Rx \lor {\sim}Px)$	
3. ${\sim}(Rx \lor {\sim}Px)$	2, UI
4. ${\sim}Rx \bullet {\sim}{\sim}Px$	3, DM
5. ${\sim}Rx$	4, simp
6. ${\sim}Rx \bullet Px$	4, DN
7. $Px \bullet {\sim}Rx$	6, com
8. Px	7, simp
9. $Px \supset Qx$	1, UI
10. Qx	9, 8, MP
11. $Qx \bullet {\sim}Rx$	10, 5, Conj
12. $(\exists x)(Qx \bullet {\sim}Rx)$	11, EG

QED

39.

1. $(\forall x)[Ax \supset (Bx \lor Cx)]$	
2. $(\exists x){\sim}(Bx \lor {\sim}Ax)$	/ $(\exists x)Cx$
3. ${\sim}(Ba \lor {\sim}Aa)$	2, EI
4. ${\sim}Ba \bullet {\sim}{\sim}Aa$	3, DM
5. ${\sim}{\sim}Aa \bullet {\sim}Ba$	4, Com
6. ${\sim}{\sim}Aa$	5, Simp
7. Aa	6, DN
8. $Aa \supset (Ba \lor Ca)$	1, UI
9. $Ba \lor Ca$	8, 7, MP
10. ${\sim}Ba$	4, Simp
11. Ca	9, 10, DS
12. $(\exists x)Cx$	11, EG

QED

40.

1. $(\exists x)[(Sx \lor Tx) \bullet Ux]$	
2. $(\forall x)(Ux \supset {\sim}Sx)$	
3. $(Sa \lor Ta) \bullet Ua$	1, EI
4. $Ua \supset {\sim}Sa$	2, UI
5. $Ua \bullet (Sa \lor Ta)$	3, com
6. Ua	5, simp
7. ${\sim}Sa$	4, 6 MP
8. $(\exists x){\sim}Sx$	7, EG
9. $Ua \bullet (Sa \lor Ta)$	3, com
10. $(Ua \bullet Sa) \lor (Ua \bullet Ta)$	9, dist
11. ${\sim}Sa \lor {\sim}Ua$	7, add
12. ${\sim}Ua \lor {\sim}Sa$	11, com
13. ${\sim}(Ua \bullet Sa)$	12, DM
14. $Ua \bullet Ta$	10, 13 DS
15. $(\exists y)(Uy \bullet Ty)$	14, EG
16. $(\exists x){\sim}Sx \bullet (\exists y)(Uy \bullet Ty)$	8, 15, conj

QED

41.

1. (∀x)(Sx ∨ Tx)	
2. (∀x)~(Ux ⊃ Sx)	
3. Sx ∨ Tx	1, UI
4. ~(Ux ⊃ Sx)	2, UI
5. ~(~Ux ∨ Sx)	4, impl
6. ~~Ux • ~Sx	5, DM
7. ~Sx • ~~Ux	6, com
8. ~Sx	7, simp
9. Tx	3, 8 DS
10. (∀x)Tx	9, UG
11. ~(Ua ⊃ Sa)	2, UI
12. ~(~Ua ∨ Sa)	11, impl
13. ~~Ua • ~Sa	12, DM
14. Ua • ~Sa	13, Dn
15. Ua	14, simp
16. (∃y)Uy	15, EG
17. (∀x)Tx • (∃y)Uy	10, 16 conj
QED	

42.

1. (∀x)(Hx ⊃ ~Jx)	
2. (∀x)(Ix ⊃ Jx)	
3. Ha • Ib	
4. Ha ⊃ ~Ja	1, UI
5. Ia ⊃ Ja	2, UI
6. ~Ja ⊃ ~Ia	5, cont
7. Ha ⊃ ~Ia	4, 6 HS
8. Ha	3, simp
9. ~Ia	7, 8 MP
10. Hb ⊃ ~Jb	1, UI
11. Ib ⊃ Jb	2, UI
12. ~Jb ⊃ ~Ib	11, cont
13. Hb ⊃ ~Ib	10, 12 HS
14. Ib • Ha	3, com
15. Ib	14, simp
16. ~~Ib 15, DN	
17. ~Hb	13, 16 MT
18. ~Ia • ~Hb	9, 17 conj
19. ~(Ia ∨ Hb)	18, DM
QED	

43.

1. (∃x)Ax	
2. (∀x)(Ax ⊃ Bx)	
3. (∀x)~(Ex ⊃ Bx)	/ (∃x)Cx
4. Aa	1, EI
5. Aa ⊃ Ba	2, UI
6. Ba	5, 4, MP
7. ~(Ea ⊃ Ba)	3, UI
8. ~(~Ea ∨ Ba)	7, Impl
9. ~~Ea • ~Ba	8, DM
10. ~Ba • ~~Ea	9, Com
11. ~Ba	10, Simp
12. Ba ∨ (∃x)Cx	6, Add
13. (∃x)Cx	12, 11, DS

QED

44.

1. (∃x)(~Tx • Ux) ≡ (∀x)Wx	
2. (∀x)(Tx ⊃ Vx)	
3. (∃x)(Ux • ~Vx)	
4. Ua • ~Va	3, EI
5. Ta ⊃ Va	2, UI
6. Ua	4, simp
7. ~Va • Ua	4, com
8. ~Va	7, simp
9. ~Ta	5, 8 MT
10. ~Ta • Ua	9, 6, conj
11. (∃x)(~Tx • Ux)	10, EG
12. [(∃x)(~Tx • Ux) ⊃ (∀x)Wx] • [(∀x)Wx ⊃ (∃x)(~Tx • Ux)]	1, equiv
13. (∃x)(~Tx • Ux) ⊃ (∀x)Wx	12, simp
14. (∀x)Wx	13, 11 MP

QED

45.

1. (∃x)(Jx ≡ Kx) ⊃ (∀x)(Ix • Lx)	
2. (∀x)[(Ix • Jx) ⊃ Kx]	
3. (∃x)~(Ix ⊃ Kx)	
4. ~(Ia ⊃ Ka)	3, EI
5. (Ia • Ja) ⊃ Ka	2, UI
6. ~(~Ia ∨ Ka)	4, impl
7. ~~Ia • ~Ka	6, DM
8. Ia • ~Ka	7, DN
9. Ia	8, simp
10. Ia ⊃ (Ja ⊃ Ka)	5, exp
11. Ja ⊃ Ka	9, 10 MP
12. ~Ka • Ia	8, com
13. ~Ka	12, simp
14. ~Ka ∨ Ja	13, add
15. Ka ⊃ Ja	14, impl
16. (Ja ⊃ Ka) • (Ka ⊃ Ja)	11, 15 conj
17. Ja ≡ Ka	16, equiv
18. (∃x)(Jx ≡ Kx)	17, EG
19. (∀x)(Ix • Lx)	1, 18 MP
20. Ix • Lx	19, UI
21. Lx • Ix	20, com
22. Lx	21, simp
23. (∀y)Ly	22, UG

QED

46.

1. (∃x)Kx ⊃ (∀x)(Lx ⊃ Mx)	
2. (∀x)~(Kx ⊃ ~Lx)	
3. (∀x)~Mx	/ (∃x)~Lx
4. ~(Ka ⊃ ~La)	2, UI
5. ~(~Ka ∨ ~La)	4, impl
6. ~~(Ka • La)	5, DM
7. Ka • La	6, DN
8. Ka	7, Simp
9. (∃x)Kx	8, EG
10. (∀x)(Lx ⊃ Mx)	1, 9, MP
11. Lx ⊃ Mx	10, UI
12. ~Mx	3, UI
13. ~Lx	11, 12, MT
14. (∃x)~Lx	13, EI

QED

47. 1. (∃x)[Ix ∨ (Hx ∨ Jx)]

2. $(\forall x)\sim(\sim Ix \supset Jx)$	
3. $(\forall x)\sim(Hx \bullet Kx)$	/ $(\exists x)\sim Kx$
4. $Ia \vee (Ha \vee Ja)$	1, EI
5. $\sim(\sim Ia \supset Ja)$	2, UI
6. $\sim(\sim\sim Ia \vee Ja)$	5, Impl
7. $\sim\sim\sim Ia \bullet \sim Ja$	6, DM
8. $\sim Ia \bullet \sim Ja$	7, DN
9. $\sim Ia$	8, Simp
10. $Ha \vee Ja$	4, 9, DS
11. $\sim Ja \bullet \sim Ia$	8, Com
12. $\sim Ja$	11, Simp
13. $Ja \vee Ha$	10, Com
14. Ha	13, 12, DS
15. $\sim(Ha \bullet Ka)$	3, UI
16. $\sim Ha \vee \sim Ka$	15, DM
17. $\sim\sim Ha$	14, DN
18. $\sim Ka$	16, 17, DS
19. $(\exists x)\sim Kx$	18, EG

QED

Exercises 3.4

1.
1. (∀x)Ax ⊃ (∃x)Bx
2. (∀x)~Bx
3. ~(∃x)Bx 2, QE
4. ~(∀x)Ax 1, 3 MT
5. (∃x)~Ax 4, QE
6. ~Ab 5, EI

QED

2.
1. (∃x)[Qx • (Rx • ~Sx)]
2. Qa • (Ra • ~Sa) 1, EI
3. (Qa • Ra) • ~Sa 2, assoc
4. ~Sa • (Qa • Ra) 3, com
5. ~Sa 4, simp
6. (∃y)~Sy 5, EG
7. ~(∀y)Sy 6, QE

QED

3.
1. (∀x)(Jx • Kx) ∨ ~(∀x)Lx
2. ~Ja
3. ~Ja ∨ ~Ka 2, add
4. ~(Ja • Ka) 3, DM
5. (∃x)~(Jx • Kx) 4, EG
6. ~(∀x)(Jx • Kx) 5, QE
7. ~(∀x)Lx 1,6 DS
8. (∃x)~Lx 7, QE

QED

4.
1. (∃x)~Ix ⊃ (∀x)(Jx ∨ Kx)
2. ~(∀x)Ix • ~Jb
3. ~(∀x)Ix 2, simp
4. (∃x)~Ix 3, QE
5. (∀x)(Jx ∨ Kx) 1, 4 MP
6. Jb ∨ Kb 5, UI
7. ~Jb • ~(∀x)Ix 2, com
8. ~Jb 7, simp
9. Kb 6, 8 DS

QED

5.
1. (∃x)Cx ∨ (∀x)Dx
2. (∀x)~(Cx ∨ Ex)
3. ~(Cx ∨ Ex) 2, UI
4. ~Cx • ~Ex 3, DM
5. ~Cx 4, simp
6. (∀x)~Cx 5, UG
7. ~(∃x)Cx 6, QE
8. (∀x)Dx 1, 7 DS

QED

6.

1. $\sim(\exists x)(Rx \vee Sx) \vee (\forall x)(Tx \supset \sim Rx)$	
2. Ra	
3. $(\exists x)(Rx \vee Sx) \supset (\forall x)(Tx \supset \sim Rx)$	1, impl
4. $Ra \vee Sa$	2, add
5. $(\exists x)(Rx \vee Sx)$	4, EG
6. $(\forall x)(Tx \supset \sim Rx)$	3, 5 MP
7. $Ta \supset \sim Ra$	6, UI
8. $\sim\sim Ra$	2, DN
9. $\sim Ta$	7, 8 MT
10. $(\exists x)\sim Tx$	9, EG
11. $\sim(\forall x)Tx$	10, QE

QED

7.

1. $(\exists x)\sim Fx \vee (\forall x)(Gx \bullet Hx)$	
2. $(\forall x)[(Fx \bullet Gx) \vee (Fx \bullet Hx)]$	
3. $\sim(\forall x)Fx \vee (\forall x)(Gx \bullet Hx)$	1, QE
4. $(\forall x)Fx \supset (\forall x)(Gx \bullet Hx)$	3, impl
5. $(Fx \bullet Gx) \vee (Fx \bullet Hx)$	2, UI
6. $Fx \bullet (Gx \vee Hx)$	5, dist
7. Fx	6, simp
8. $(\forall x)Fx$	7, UG
9. $(\forall x)(Gx \bullet Hx)$	4, 8 MP
10. $Ga \bullet Ha$	9, UI
11. $(\exists y)(Gy \bullet Hy)$	10, EG

QED

8.

1. $\sim(\forall x)(Qx \supset Rx)$	
2. $(\forall x)(\sim Rx \supset Tx)$	/ $\sim(\forall x)\sim Tx$
3. $(\exists x)\sim(Qx \supset Rx)$	1, QE
4. $\sim(Qa \supset Ra)$	3, EI
5. $\sim(\sim Qa \vee Ra)$	4, Impl
6. $\sim\sim Qa \bullet \sim Ra$	5, DM
7. $\sim Ra \bullet \sim\sim Qa$	6, Com
8. $\sim Ra$	7, Simp
9. $\sim Ra \supset Ta$	2, UI
10. Ta	9, 8, MP
11. $(\exists x)Tx$	10, EG
12. $\sim(\forall x)\sim Tx$	11, QE

QED

9.

1. $(\forall x)[Lx \vee (Mx \bullet \sim Nx)]$	
2. $\sim(\exists x)Lx$	/ $\sim(\exists x)(Lx \vee Nx)$
3. $(\forall x)\sim Lx$	2, QE
4. $Lx \vee (Mx \bullet \sim Nx)$	1, UI
5. $\sim Lx$	3, UI
6. $Mx \bullet \sim Nx$	4, 5, DS
7. $\sim Nx \bullet Mx$	6, Com
8. $\sim Nx$	7, Simp
9. $\sim Lx \bullet \sim Nx$	5, Conj
10. $\sim(Lx \vee Nx)$	9, DM
11. $(\forall x)\sim(Lx \vee Nx)$	10, UG
12. $\sim(\exists x)(Lx \vee Nx)$	11, QE

QED

10.	1. (∀x)[(Tx • Ux) ⊃ Vx]	
	2. ~(∀x)~Tx	
	3. (∃x)Tx	2, QE
	4. Ta	3, EI
	5. (Ta • Ua) ⊃ Va	1, UI
	6. Ta ⊃ (Ua ⊃ Va)	5, exp
	7. Ua ⊃ Va	6, 4, MP
	8. ~Ua ∨ Va	7, impl
	9. ~Ua ∨ ~~Va	8, DN
	10. ~(Ua • ~Va)	9, DM
	11. (∃x)~(Ux • ~Vx)	10, EG
	12. ~(∀x)(Ux • ~Vx)	11, QE
	QED	

11.	1. (∀x)(Ax ∨ Bx)	
	2. (∀x)(Ax ⊃ Dx)	
	3. ~(∀x)(Bx • ~Cx)	
	4. (∃x)~(Bx • ~Cx)	3, QE
	5. ~(Ba • ~Ca)	4, EI
	6. Aa ∨ Ba	1, UI
	7. Aa ⊃ Da	2, UI
	8. ~Ba ∨ ~~Ca	5, DM
	9. ~Ba ∨ Ca	8, DN
	10. Ba ⊃ Ca	9, impl
	11. (Aa ⊃ Da) • (Ba ⊃ Ca)	7, 10 conj
	12. Da ∨ Ca	11, 6, CD
	13. (∃y)(Dy ∨ Cy)	12, EG
QED		

12.	1. ~(∀x)[Kx ⊃ (Lx ⊃ Mx)]	
	2. (∀x)[(Nx • Ox) ≡ Mx]	
	3. (∃x)~[Kx ⊃ (Lx ⊃ Mx)]	1, QE
	4. ~[Ka ⊃ (La ⊃ Ma)]	3, EI
	5. ~[(Ka • La) ⊃ Ma]	4, exp
	6. ~[~(Ka • La) ∨ Ma]	5, impl
	7. ~~(Ka • La) • ~Ma	6, DM
	8. ~Ma • ~~(Ka • La)	7, com
	9. ~Ma	8, simp
	10. (Na • Oa) ≡ Ma	2, UI
	11. [(Na • Oa) ⊃ Ma] • [Ma ⊃ (Na • Oa)]	10, equiv
	12. (Na • Oa) ⊃ Ma	11, simp
	13. ~(Na • Oa)	9, 12 MT
	14. (∃x)~(Nx • Ox)	13, EG
	15. ~(∀x)(Nx • Ox)	14, QE
QED		

13. 1. ~(∃x)(Ox ≡ Px)
2. Pa
3. (∀x)~(Ox ≡Px) 1, QE
4. ~(Oa ≡ Pa) 3, EI
5. ~[(Oa ⊃ Pa) • (Pa ⊃ Oa)] 4, equiv
6. ~(Oa ⊃ Pa) ∨ ~(Pa ⊃ Oa) 5, DM
7. (Oa ⊃ Pa) ⊃ ~(Pa ⊃ Oa) 6, impl
8. Pa ∨ ~Oa 2, add
9. ~Oa ∨ Pa 8, com
10. Oa ⊃ Pa 9, impl
11. ~(Pa ⊃ Oa) 7, 10 MP
12. ~(~Pa ∨ Oa) 11, impl
13. ~~Pa • ~Oa 12, DM
14. ~Oa • ~~Pa 13, com
15. ~Oa 14, simp
16. (∃x)~Ox 15, EG
17. ~(∀x)Ox 16, QE
QED

14. 1. ~(∃x)[Ex • (Fx ∨ Gx)]
2. (∀x)[Hx ⊃ (Ex • Gx)]
3. (∃x)[~Hx ⊃ (Ix ∨ Jx)]
4. ~Ha ⊃ (Ia ∨ Ja) 3, EI
5. (∀x)~[Ex • (Fx ∨ Gx)] 1, QE
6. ~[Ea • (Fa ∨ Ga)] 5, UI
7. Ha ⊃ (Ea • Ga) 2, UI
8. ~Ea ∨ ~(Fa ∨ Ga) 6, DM
9. ~Ea ∨ (~Fa • ~Ga) 8, DM
10. (~Ea ∨ ~Fa) • (~Ea ∨ ~Ga) 9, dist
11. (~Ea ∨ ~Ga) • (~Ea ∨ ~Fa) 10, com
12. ~Ea ∨ ~Ga 11, simp
13. ~(Ea • Ga) 12, DM
14. ~Ha 7, 13 MT
15. Ia ∨ Ja 4, 14 MP
16. ~~Ia ∨ Ja 15, DN
17. ~Ia ⊃ Ja 16, impl
18. (∃x)(~Ix ⊃ Jx) 17, EG
QED

15. 1. ~(∃x)[Fx • (Gx • Hx)]
2. ~(∃x)(Ix • ~Fx) / (∀x)[Ix ⊃ (~Gx ∨ ~Hx)]
3. (∀x)~[Fx • (Gx • Hx)] 1, QE
4. (∀x)~(Ix • ~Fx) 2, QE
5. ~[Fx • (Gx • Hx) 3, UI
6. ~Fx ∨ ~(Gx • Hx) 5, DM
7. Fx ⊃ ~(Gx • Hx) 6, Impl
8. ~(Ix • ~Fx) 4, UI
9. ~Ix ∨ ~~Fx 8, DM
10. Ix ⊃ ~~Fx 9, Impl
11. Ix ⊃ Fx 10, DN
12. Ix ⊃ ~(Gx • Hx) 11, 7, HS
13. Ix ⊃ (~Gx ∨ ~Hx) 12, DM
14. (∀x)[Ix ⊃ (~Gx ∨ ~Hx)] 13, UG
QED

16. 1. ~(∀x)[(Jx • Kx) • Lx]
2. (∀x)(Mx ⊃ Jx)

	3. $(\forall x)(\sim Nx \bullet Mx)$	
	4. $(\exists x)\sim[(Jx \bullet Kx) \bullet Lx]$	1, QE
	5. $\sim[(Ja \bullet Ka) \bullet La]$	4, EI
	6. $Ma \supset Ja$	2, UI
	7. $\sim Na \bullet Ma$	3, UI
	8. $Ma \bullet \sim Na$	7, com
	9. Ma	8, simp
	10. Ja	6, 9 MP
	11. $\sim(Ja \bullet Ka) \vee \sim La$	5, DM
	12. $(\sim Ja \vee \sim Ka) \vee \sim La$	11, DM
	13. $\sim Ja \vee (\sim Ka \vee \sim La)$	12, assoc
	14. $Ja \supset (\sim Ka \vee \sim La)$	13, impl
	15. $\sim Ka \vee \sim La$	10, 14 MP
	16. $\sim(Ka \bullet La)$	15, DM
	17. $(\exists x)\sim(Kx \bullet Lx)$	16, EG
	18. $\sim(\forall x)(Kx \bullet Lx)$	17, QE
	QED	

17.	1. $(\exists x)[(Ax \vee Cx) \supset Bx]$	
	2. $\sim(\exists x)(Bx \vee Ex)$	
	3. $(\exists x)(Dx \supset Ex) \supset (\forall x)(Ax \vee Cx)$	
	4. $(\forall x)\sim(Bx \vee Ex)$	2, QE
	5. $(Aa \vee Ca) \supset Ba$	1, EI
	6. $\sim(Ba \vee Ea)$	4, UI
	7. $\sim Ba \bullet \sim Ea$	6, DM
	8. $\sim Ba$	7, simp
	9. $\sim(Aa \vee Ca)$	5, 8 MT
	10. $(\exists x)\sim(Ax \vee Cx)$	9, EG
	11. $\sim(\forall x)(Ax \vee Cx)$	10, QE
	12. $\sim(\exists x)(Dx \supset Ex)$	3, 11 MT
	13. $(\forall x)\sim(Dx \supset Ex)$	12, QE
	14. $\sim(Da \supset Ea)$	13, UI
	15. $\sim(\sim Da \vee Ea)$	14, impl
	16. $\sim\sim Da \bullet \sim Ea$	15, DM
	17. $Da \bullet \sim Ea$	16, DN
	18. Da	17, simp
	19. $(\exists y)Dy$	18, EG
	QED	

18.	1. (∃x)(Nx ∨ ~Ox)	
	2. ~(∀x)(Px • Qx) • ~(∃x)(Nx ∨ ~Qx)	
	3. ~(∃x)(Nx ∨~ Qx) • ~(∀x)(Px • Qx)	2, com
	4. ~(∃x)(Nx ∨ ~Qx)	3, simp
	5. (∀x)~(Nx ∨ ~Qx)	4, QE
	6. Na ∨ ~Oa	1, EI
	7. ~(Na ∨ ~Qa)	5, UI
	8. ~Na • ~~Qa	7, DM
	9. ~Na	8, simp
	10. ~Oa	6, 9 DS
	11. ~(∀x)(Px • Qx)	2, simp
	12. (∃x)~(Px • Qx)	11, QE
	13. ~(Pb • Qb)	12, EI
	14. ~Pb ∨ ~Qb	13, DM
	15. ~(Nb ∨ ~Qb)	5, UI
	16. ~Nb • ~~Qb	15, DM
	17. ~~Qb • ~Nb	16, com
	18. ~~Qb	17, simp
	19. ~Qb ∨ ~Pb	14, com
	20. ~Pb	18, 19 DS
	21. (∃x)~Px	20, EG
	22. ~(∀x)Px	21, QE
	23. (∃x)~Ox	10, EG
	24. ~(∀x)Ox	23, QE
	25. ~(∀x)Px • ~(∀x)Ox	22, 24 conj
	26. ~[(∀x)Px ∨ (∀x)Ox]	25, DM
	QED	

19.	1. (∃x)(Mx • ~Nx) ⊃ (∀x)(Ox ∨ Px)	
	2. ~(∀x)(~Nx ⊃ Ox)	
	3. ~(∃x)Px	
	4. (∃x)~(~Nx ⊃ Ox)	2, QE
	5. ~(~Na ⊃ Oa)	4, EI
	6. ~(~~Na ∨ Oa)	5, DM
	7. ~(Na ∨ Oa)	6, DN
	8. ~Na • ~Oa	7, DM
	9. ~Oa • ~Na	8, com
	10. ~Oa	9, simp
	11. (∀x)~Px	3, QE
	12. ~Pa	11, UI
	13. ~Oa • ~Pa	10, 12 conj
	14. ~(Oa ∨ Pa)	13, DM
	15. (∃x)~(Ox ∨ Px)	14, EG
	16. ~(∀x)(Ox ∨ Px)	15, QE
	17. ~(∃x)(Mx • ~Nx)	1, 16 MT
	18. (∀x)~(Mx • ~Nx)	17, QE
	19. ~(Ma • ~Na)	18, UI
	20. ~Ma ∨ ~~Na	20, DM
	21. ~~Na ∨ ~Ma	20, Com
	22. Na ∨ ~Ma	21, DN
	23. ~Na	8, Simp
	24. (∃y)~Ny	23, EG
	25. ~(∀y)Ny	24, QE

QED

20.

1. (∃x)[Ax • (Bx ∨ Cx)] ⊃ (∀x)Dx
2. ~(∀x)(Ax ⊃ Dx)
3. (∃x)~(Ax ⊃ Dx) 2, QE
4. ~(Aa ⊃ Da) 3, EI
5. ~(~Aa ∨ Da) 4, impl
6. ~~Aa • ~Da 5, DM
7. ~Da • ~~Aa 6, com
8. ~Da 7, simp
9. (∃x)~Dx 8, EG
10. ~(∀x)Dx 9, QE
11. ~(∃x)[Ax • (Bx ∨ Cx)]1, 10 MT
12. (∀x)~[Ax • (Bx ∨ Cx)]11, QE
13. ~[Aa • (Ba ∨ Ca)] 12, UI
14. ~Aa ∨ ~(Ba ∨ Ca) 13, DM
15. ~Aa ∨ (~Ba • ~Ca) 14, DM
16. ~~Aa 6, simp
17. ~Ba • ~Ca 15, 16 DS
18. ~Ca • ~Ba 17, com
19. ~Ca 18, simp
20. (∃x)~Cx 19, EG
21. ~(∀x)Cx 20, QE

QED

21.

1. (∀x)(Ex • Fx) ∨ ~(∀x)[Gx ⊃ (Hx ⊃ Ix)]
2. ~(∀x)(Jx ⊃ Ex)
3. (∃x)~(Jx ⊃ Ex) 2, QE
4. ~(Ja ⊃ Ea) 3, EI
5. ~(~Ja ∨ Ea) 4, impl
6. ~~Ja • ~Ea 5, DM
7. ~Ea • ~~Ja 6, com
8. ~Ea 7, simp
9. ~Ea ∨ ~Fa 8, add
10. ~(Ea • Fa) 9, DM
11. (∃x)~(Ex • Fx) 10, EG
12. ~(∀x)(Ex • Fx) 11, QE
13. ~(∀x)[Gx ⊃ (Hx ⊃ Ix)] 1, 12 DS
14. (∃x)~[Gx ⊃ (Hx ⊃ Ix)] 13, QE
15. ~[Ga ⊃ (Ha ⊃ Ia)] 14, EI
16. ~[(Ga • Ha) ⊃ Ia] 14, exp
17. ~[~(Ga • Ha) ∨ Ia] 16, impl
18. ~~(Ga • Ha) • ~Ia 17, DM
19. ~Ia • ~~(Ga • Ha) 18, com
20. ~Ia 19, simp
21. (∃y)~Iy 20, EG
22. ~(∀y)Iy 21, QE

QED

22.

1. $\sim(\exists x)(Jx \bullet \sim Kx)$	
2. $\sim(\exists x)[Kx \bullet (\sim Jx \vee \sim Lx)]$	/ $(\forall x)(Jx \equiv Kx)$
3. $(\forall x)\sim(Jx \bullet \sim Kx)$	1, QE
4. $\sim(Jx \bullet \sim Kx)$	3, UI
5. $\sim Jx \vee \sim\sim Kx$	4, DM
6. $Jx \supset \sim\sim Kx$	5, Impl
7. $Jx \supset Kx$	6, DN
8. $(\forall x)\sim[Kx \bullet (\sim Jx \vee \sim Lx)]$	2, QE
9. $\sim[Kx \bullet (\sim Jx \vee \sim Lx)]$	8, UI
10. $\sim Kx \vee \sim(\sim Jx \vee \sim Lx)$	9, DM
11. $\sim Kx \vee (\sim\sim Jx \bullet \sim\sim Lx)$	10, DM
12. $(\sim Kx \vee \sim\sim Jx) \bullet (\sim Kx \vee \sim\sim Lx)$	11, Dist
13. $\sim Kx \vee \sim\sim Jx$	12, Simp
14. $\sim Kx \vee Jx$	13, DN
15. $Kx \supset Jx$	14, Impl
16. $(Jx \supset Kx) \bullet (Kx \supset Jx)$	7, 15, Conj
17. $Jx \equiv Kx$	16, equiv
18. $(\forall x)(Jx \equiv Kx)$	17, UG

QED

23.

1. $\sim[(\exists x)(Ax \vee Bx) \bullet (\forall x)(Cx \supset Dx)]$	
2. $\sim(\forall x)(\sim Ax \vee Ex)$	
3. $\sim(\exists x)(Ax \vee Bx) \vee \sim(\forall x)(Cx \supset Dx)$	1, DM
4. $(\exists x)(Ax \vee Bx) \supset \sim(\forall x)(Cx \supset Dx)$	3, impl
5. $(\exists x)\sim(\sim Ax \vee Ex)$	2, QE
6. $\sim(\sim Aa \vee Ea)$	5, EI
7. $\sim\sim Aa \bullet \sim Ea$	6, DM
8. $Aa \bullet \sim Ea$	7, DN
9. Aa	8, simp
10. $Aa \vee Ba$	9, add
11. $(\exists x)(Ax \vee Bx)$	10, EG
12. $\sim(\forall x)(Cx \supset Dx)$	4, 11 MP
13. $(\exists x)\sim(Cx \supset Dx)$	12, QE
14. $\sim(Cb \supset Db)$	13, UI
15. $\sim(\sim Cb \vee Db)$	14, impl
16. $\sim\sim Cb \bullet \sim Db$	15, DM
17. $\sim\sim Cb$	16, Simp
18. Cb	17, DN
19. $(\exists x)Cx$	18, EG

QED

24.

1. (∀x)(Fx ⊃ Hx) ∨ ~(∃x)(Gx ≡ Ix)	
2. (∃x)[Fx • (~Hx • Ix)]	
3. Fa • (~Ha • Ia)	2, EI
4. (Fa • ~Ha) • Ia	3, assoc
5. Fa • ~Ha	4, simp
6. ~~Fa • ~Ha	5, DN
7. ~(~Fa ∨ Ha)	6, DM
8. ~(Fa ⊃ Ha)	7, impl
9. (∃x)~(Fx ⊃ Hx)	8, EG
10. ~(∀x)(Fx ⊃ Hx)	9, QE
11. ~(∃x)(Gx ≡ Ix)	1, 10 DS
12. (∀x)~(Gx ≡ Ix)	11, QE
13. ~(Ga ≡ Ia)	12, UI
14. ~[(Ga • Ia) ∨ (~Ga • ~Ia)]	13, equiv
15. ~(Ga • Ia) • ~(~Ga • ~Ia)	14, DM
16. ~(Ga • Ia)	15, simp
17. ~Ga ∨ ~Ia	16, DM
18. ~Ia ∨ ~Ga	17, com
19. Ia • (Fa • ~Ha)	4, com
20. Ia	19, simp
21. ~~Ia	20, dn
22. ~Ga	18, 21 DS
23. (∃x)~Gx	22, EG
24. ~(∀x)Gx	23, QE

QED

25.

1. ~(∃x)[Px • (Qx • Rx)]	
2. ~(∀x)[~Rx ∨ (Sx •Tx)]	
3. (∀x)(Px • Qx) ∨ (∀x)(Tx ⊃ Rx)	
4. (∀x)~[Px • (Qx • Rx)]	1, QE
5. (∃x)~[~Rx ∨ (Sx •Tx)]	2, QE
6. ~[~Ra ∨ (Sa • Ta)]	5, EI
7. ~[Pa • (Qa • Ra)]	4, UI
8. ~Pa ∨ ~(Qa • Ra)	7, DM
9. ~Pa ∨ (~Qa ∨ ~Ra)	8, DM
10. Pa ⊃ (~Qa ∨ ~Ra)	9, impl
11. Pa ⊃ (Qa ⊃ ~Ra)	10, impl
12. (Pa • Qa) ⊃ ~Ra	11, exp
13. ~~Ra • ~(Sa • Ta)	6, DM
14. ~~Ra	13, simp
15. ~(Pa • Qa)	12, 14 MT
16. (∃x)~(Px • Qx)	15, EG
17. ~(∀x)(Px • Qx)	16, QE
18. (∀x)(Tx ⊃ Rx)	3, 17 DS
19. (∀x)(~Tx ∨ Rx)	18, Impl
20. (∀x)(~Tx ∨ ~~Rx)	19, DN
21. (∀x)~(Tx • ~Rx)	20, DM
22. ~(∃x)(Tx • ~Rx)	21, QE

QED

Exercises 3.5

1.

1. (∀x)(Dx ∨ Ex)		
2. (∀x)(Fx ⊃ ~Ex)		
	| 3. ~Dx	ACP
	| 4. Dx ∨ Ex	1, UI
	| 5. Fx ⊃ ~Ex	2, UI
	| 6. Ex	3, 4 DS
	| 7. ~~Ex	6, DN
	| 8. ~Fx	5,7 MT
9. ~Dx ⊃ ~Fx		3-8 CP
10. (∀x)(~Da ⊃ ~Fa)		9, UG

QED

2.

1. (∀x)(Ax ⊃ Bx)		
2. (∀x)~(Bx • ~Cx)		
	| 3. Ax	ACP
	| 4. Ax ⊃ Bx	1, UI
	| 5. ~(Bx • ~Cx)	2, UI
	| 6. Bx	3,4 MP
	| 7. ~Bx ∨ ~~Cx	5, DM
	| 8. ~~Bx	6, DN
	| 9. ~~Cx	7,8 DS
	| 10. Cx	9, DN
11. Ax ⊃ Cx		3-10 CP
12. (∀x)(Ax ⊃ Cx)		11, UG

QED

3.

1. (∀x)(Gx ⊃ Hx)		
2. ~(∃x)(Ix • ~Gx)		
3. (∀x)(~Hx ⊃ Ix)		
	| 4. (∃x)~Hx	AIP
	| 5. ~Ha	4, EI
	| 6. Ga ⊃ Ha	1, UI
	| 7. ~Ga	6, 5, MT
	| 8. (∀x)~(Ix • ~Gx)	2, QE
	| 9. ~(Ia • ~Ga)	8, UI
	| 10. ~Ia ∨ ~~Ga	9, DM
	| 11. ~~Ga ∨ ~Ia	10, Com
	| 12. Ga ∨ ~Ia	11, DN
	| 13. ~Ia	12, 7, DS
	| 14. ~Ha ⊃ Ia	3, UI
	| 15. ~~Ha	14, 13, MT
	| 16. ~Ha • ~~Ha	5, 15, Conj
17. ~(∃x)~Hx		4-16, IP
18. (∀x)Hx		17, QE

QED

4.

1. (∀x)[Ax ⊃ (Bx ⊃ Cx)]	
2. ~(∀x)(Bx ⊃ Dx)	
│3. (∀x)Ax	ACP
│4. (∃x)~(Bx ⊃ Dx)	2, QE
│5. ~(Ba ⊃ Da)	4, EI
│6. ~(~Ba ∨ Da)	5, impl
│7. ~~Ba • ~Da	6, DM
│8. Ba • ~Da	7, DN
│9. Ba	8, simp
│10. Aa	3, UI
│11. Aa ⊃ (Ba ⊃ Ca)	1, UI
│12. Ba ⊃ Ca	10, 11 MP
│13. Ca	9, 12 MP
│14. ~Da • Ba	8, Com
│15. ~Da	14, Simp
│16. Ca • ~Da	13, 15, Conj
│17. (∃x)(Cx • ~Dx)	16, EG
18. (∀x)Ax ⊃ (∃x)(Cx • ~Dx)	3-17 CP

QED

5.

1. (∀x)(Rx ⊃ Ux)	
2. ~(∃x)(Ux • Sx)	
│3. (∃x)Rx	ACP
│4. Ra	ACP
│5. (∀x)~(Ux • Sx)	2, QE
│6. ~(Ua • Sa)	5, UI
│7. ~Ua ∨ ~Sa	6, DM
│8. Ra ⊃ Ua	1, UI
│9. Ua	8, 4, MP
│10. ~~Ua	9, DN
│11. ~Sa	7, 10, DS
│12. (∃x)~Sx	11, EG
13. (∃x)Rx ⊃ (∃x)~Sx	3-12 CP

QED

6.

1. (∀x)[Ax ⊃ (Dx ∨ Ex)]	
2. (∀x)[(~Dx ⊃ Ex) ⊃ (~Cx ⊃ Bx)]	
│3. Ax	ACP
│4. Ax ⊃ (Dx ∨ Ex)	1, UI
│5. Dx ∨ Ex	3, 4 MP
│6. ~~Dx ∨ Ex	5, DN
│7. ~Dx ⊃ Ex	6, impl
│8. (~Dx ⊃ Ex) ⊃ (~Cx ⊃ Bx)	2, UI
│9. ~Cx ⊃ Bx	7, 8 MP
│10. ~Bx ⊃ ~~Cx	9, cont
│11. ~Bx ⊃ Cx	10, DN
│12. ~~Bx ∨ Cx	11, impl
│13. Bx ∨ Cx	12, DN
14. Ax ⊃ (Bx ∨ Cx)	3-13 CP
15. (∀x)[Ax ⊃ (Bx ∨ Cx)]	14, UG

QED

7.
1. (∀x)[~Nx ∨ (Qx • Rx)]
2. (∀x)(Px ≡ Qx)
| 3. (∃x)Nx — ACP
| 4. Na — 3, EI
| 5. ~Na ∨ (Qa • Ra) — 1, UI
| 6. ~~Na — 4, DN
| 7. Qa • Ra — 5, 6 DS
| 8. Qa — 7, simp
| 9. Pa ≡ Qa — 2, UI
| 10. (Pa ⊃ Qa) • (Qa ⊃ Pa) — 9, equiv
| 11. (Qa ⊃ Pa) • (Pa ⊃ Qa) — 10, com
| 12. Qa ⊃ Pa — 11, simp
| 13. Pa — 8, 12 MP
| 14. (∃x)Px — 13, EG
15. (∃x)Nx ⊃ (∃x)Px — 3-14, CP
QED

8.
1. (∀x)(Px ⊃ Qx)
2. ~(∃x)[(Px • Rx) • Qx]
3. (∃x)Rx — / ~(∀x)Px
4. Ra — 3, EI
| 5. (∀x)Px — AIP
| 6. Pa — 5, UI
| 7. Pa ⊃ Qa — 1, UI
| 8. Qa — 7, 6, MP
| 9. Pa • Ra — 8, 4, Conj
| 10. (Pa • Ra) • Qa — 9, 8, Conj
| 11. (∃x)[(Pa • Ra) • Qa] — 10, EG
| 12. (∃x)[(Pa • Ra) • Qa] • ~(∃x)[(Pa • Ra) • Qa] — 11, 2, Conj
13. ~(∀x)Px — 5-12, IP
QED

9.
1. (∀x)(Ox ⊃ Nx)
2. (∀x)(Nx ⊃ Px)
3. ~(∃x)(Px ∨ Qx)
| 4. Ox — AIP
| 5. Ox ⊃ Nx — 1, UI
| 6. Nx ⊃ Px — 2, UI
| 7. Ox ⊃ Px — 5, 6 HS
| 8. Px — 4, 7 MP
| 9. (∀x)~(Px ∨ Qx) — 3, QE
| 10. ~(Px ∨ Qx) — 8, UI
| 11. ~Px • ~Qx — 9, DM
| 12. ~Px — 11, simp
| 13. Px • ~Px — 8, 12 conj
14. ~Ox — 4-13 IP
15. (∀x)~Ox — 14, UG
QED

10.	1. (∀x)[(Fx ∨ Gx) ⊃ Ix]	
	2. (∀x)[(Ix • Ex) ⊃ Gx]	
	│3. Ex • Fx	ACP
	│4. (Fx ∨ Hx) ⊃ Ix	1, UI
	│5. Fx • Ex	3, com
	│6. Fx	5, simp
	│7. Fx ∨ Hx	6, add
	│8. Ix	4, 7 MP
	│9. (Ix • Ex) ⊃ Gx	2, UI
	│10. Ex	3, simp
	│11. Ix • Ex	8, 10 conj
	│12. Gx	9, 11 MP
	13. (Ex • Fx) ⊃ Gx	3-12 CP
	14. Ex ⊃ (Fx ⊃ Gx)	13, exp
	15. (∀x)[Ex ⊃ (Fx ⊃ Gx)]	14, UG

QED

11.	1. (∀x)[Sx ⊃ (~Tx ∨ ~Rx)]	
	2. (∀x)(Ux ⊃ Sx)	/ (∃x)(Rx • Tx) ⊃ (∃x)(~Sx • ~Ux)
	│3. (∃x)(Rx • Tx)	ACP
	│4. Ra • Ta	3, EI
	│5. Ta • Ra	4, Com
	│6. ~~(Ta • Ra)	5, DN
	│7. ~(~Ta ∨ ~Ra)	6, DM
	│8. Sa ⊃ (~Ta ∨ ~Ra)	1, UI
	│9. ~Sa	8, 7, MT
	│10. Ua ⊃ Sa	2, UI
	│11. ~Ua	10, 9, MT
	│12. ~Sa • ~Ua	9, 11, Conj
	│13. (∃x)(~Sx • ~Ux)	12, EG
	14. (∃x)(Rx • Tx) ⊃ (∃x)(~Sx • ~Ux)	3-13, CP

QED

12.	1. (∀x)(Ex ≡ Hx)	
	2. (∀x)(Hx ⊃ ~Fx)	
	│3. (∀x)Ex	ACP
	│4. Ex	3, UI
	│5. Ex ≡ Hx	1, UI
	│6. (Ex ⊃ Hx) • (Hx ⊃ ~Fx)	5, equiv
	│7. Ex ⊃ Hx	6, simp
	│8. Hx ⊃ ~Fa	2, UI
	│9. Ex ⊃ ~Fx	7, 8, HS
	│10. ~Fx	9, 4, MP
	│11. (∀x)~Fx	10, EG
	│12. ~(∃x)Fx	11, QE
	13. (∀x)Ex ⊃ ~(∃x)Fx	3-11, CP

QED

13.	1. (∀x)(Cx ⊃ Ax)	
	2. (∃x)~Bx ⊃ (∀x)Cx	
	3. ~(Aa ∨ Ba)	AIP
	4. ~Aa • ~Ba	3, DM
	5. ~Aa	4, simp
	6. Ca ⊃ Aa	1, UI
	7. ~Ca	5,6 MT
	8. ~Ba • ~Aa	4, com
	9. ~Ba	8, simp
	10. (∃x)~Bx	9, EG
	11. (∀x)Cx	2, 10 MP
	12. Ca	11, UI
	13. ~Ca • Ca	7, 12 conj
	14. ~~(Aa ∨ Ba)	3-13 IP
	15. Aa ∨ Ba	14, DN
	16. (∃x)(Ax ∨ Bx)	17, EG
	QED	

14.	1. (∀x)[Jx ⊃ (~Kx ⊃ ~Lx)	
	2. (∃x)(Jx • ~Kx)	/ ~(∀x)Lx
	3. Ja • ~Ka	2, EI
	4. Ja ⊃ (~Ka ⊃ ~La)	1, UI
	5. Ja	3, Simp
	6. ~Ka ⊃ ~La	4, 5, MP
	7. ~Ka • Ja	3, Com
	8. ~Ka	7, Simp
	9. ~La	6, 8, MP
	10. (∃x)~Lx	9, EG
	11. ~(∀x)Lx	10, QE
QED		

15.	1. (∀x)[Jx ⊃ (Mx • Lx)]	
	2. (∀x)[(~Kx ∨ Nx) • (~Kx ∨ Lx)]	/ (∀x)[(Jx ∨ Kx) ⊃ Lx]
	3. Jx ∨ Kx	ACP
	4. Jx ⊃ (Mx • Lx)	1, UI
	5. (~Kx ∨ Nx) • (~Kx ∨ Lx)	7, 8 HS
	6. ~Kx ∨ (Nx • Lx)	5, Dist
	7. Kx ⊃ (Nx • Lx)	6, Impl
	8. [Jx ⊃ (Mx • Lx)] • [Kx ⊃ (Nx • Lx)]	4, 7, Conj
	9. (Mx • Lx) ∨ (Nx • Lx)	8, 3, CD
	10. (Lx • Mx) ∨ (Nx • Lx)	9, Com
	11. (Lx • Mx) ∨ (Lx • Nx)	10, Com
	12. Lx • (Mx ∨ Nx)	11, Dist
	13. Lx	12, Simp
	14. (Jx ∨ Kx) ⊃ Lx	3-13, CP
	15. (∀x)[(Jx ∨ Kx) ⊃ Lx]	14, UG
QED		

16.
1. $(\forall x)(Ix \supset Kx)$
2. $(\forall x)(Lx \supset Jx)$
3. $\sim(\exists x)(\sim Kx \supset Jx)$

4. $(\exists x)[Ix \vee (Lx \bullet Mx)]$	AIP
5. $Ia \vee (La \bullet Ma)$	4, EI
6. $Ia \supset Ka$	1, UI
7. $La \supset Ja$	2, UI
8. $(Ia \vee La) \bullet (Ia \vee Ma)$	5, dist
9. $Ia \vee La$	8, simp
10. $(Ia \supset Ka) \bullet (La \supset Ja)$	6, 7 conj
11. $Ka \vee Ja$	10, 9 CD
12. $(\forall x) \sim(\sim Kx \supset Jx)$	3, QE
13. $\sim(\sim Ka \supset Ja)$	12, UI
14. $\sim(\sim\sim Ka \vee Ja)$	13, impl
15. $\sim(Ka \vee Ja)$	14, DN
16. $(Ka \vee Ja) \bullet \sim(Ka \vee Ja)$	11, 15 conj

17. $\sim(\exists x)[Ix \vee (Lx \bullet Mx)]$ 4-16, IP

QED

17.
1. $(\forall x)(Px \supset Ox)$
2. $(\forall x)(Ox \equiv Qx)$

3. $\sim(\sim Px \vee Qx)$	AIP
4. $\sim\sim Px \bullet \sim Qx$	3, DM
5. $Px \bullet \sim Qx$	4, DN
6. Px	5, simp
7. $Px \supset Ox$	1, UI
8. Ox	6, 7 MP
9. $Ox \equiv Qx$	2, UI
10. $(Ox \supset Qx) \bullet (Qx \supset Ox)$	9, equiv
11. $Ox \supset Qx$	10, simp
12. Qx	8, 11 MP
13. $\sim Qx \bullet Px$	5, com
14. $\sim Qx$	13, simp
15. $Qx \bullet \sim Qx$	12, 14 conj

16. $\sim\sim(\sim Px \vee Qx)$ 3-15 IP
17. $\sim Px \vee Qx$ 16, DN
18. $(\forall x)(\sim Px \vee Qx)$ 17, UI

QED

18.

1. $(\forall x)[Fx \supset (Dx \bullet \sim Ex)]$	
2. $(\forall x)(Fx \supset Hx)$	
3. $(\exists x)Fx$	/ $\sim(\forall x)(Dx \supset Ex) \vee (\exists x)[Fx \bullet (Gx \bullet Hx)]$
4. Fa	3, EI
5. $Fa \supset Ha$	2, UI
6. Ha	5, 4, MP
7. $Fa \supset (Da \bullet \sim Ea)$	1, UI
8. $Da \bullet \sim Ea$	7, 4, MP
9. $\sim Ea \bullet Da$	8, Com
10. $\sim Ea$	9, Simp
| 11. $(\forall x)(Dx \supset Ex)$	AIP
| 12. $Da \supset Ea$	3, UI
| 13. Da	8, Simp
| 14. Ea	4, 5, MP
| 15. $Ea \bullet \sim Ea$	14, 10, Conj
16. $\sim(\forall x)(Dx \supset Ex)$	11-15, IP
17. $\sim(\forall x)(Dx \supset Ex) \vee (\exists x)[Fx \bullet (Gx \bullet Hx)]$	16, Add

QED

19.

1. $(\exists x)(Sx \vee Tx)$	
2. $(\exists x)(Ux \supset \sim Vx)$	
3. $(\exists x)Tx \supset (\forall x)Ux$	
| 4. $(\forall x)(\sim Sx \bullet Vx)$	AIP
| 5. $Sa \vee Ta$	1, EI
| 6. $\sim Sa \bullet Va$	4, UI
| 7. $\sim Sa$	6, simp
| 8. Ta	5, 7 DS
| 9. $Ub \supset \sim Vb$	2, EI
| 10. $\sim Sb \bullet Vb$	4, UI
| 11. $Vb \bullet \sim Sb$	10, com
| 12. Vb	11, simp
| 13. $\sim\sim Vb$	12, DN
| 14. $\sim Ub$	9, 13 MT
| 15. $(\exists x)\sim Ux$	14, EG
| 16. $\sim(\forall x)Ux$	15, QE
| 17. $\sim(\exists x)Tx$	3, 16 MT
| 18. $(\exists x)Tx$	8, EG
| 19. $(\exists x)Tx \bullet \sim(\exists x)Tx$	18, 17, Conj
20. $\sim(\forall x)(\sim Sx \bullet Vx)$	4-19, IP

QED

20.
1. (∀x)[Ax ⊃ (Cx • Dx)]
2. (∃x)(Bx • ~Cx) / ~(∀x)(Ax ≡ Bx)
3. Ba • ~Ca 2, EI
4. ~Ca • Ba 3, Com
5. ~Ca 4, Simp
6. ~Ca ∨ ~Da 5, Add
7. ~(Ca • Da) 6, DM
8. Aa ⊃ (Ca • Da) 1, UI
9. ~Aa 8, 7, Mt
| 10. (∀x)(Ax ≡ Bx) AIP
| 11. Aa ≡ Ba 10, UI
| 12. (Aa ⊃ Ba) • (Ba ⊃ Aa) 11, Equiv
| 13. (Ba ⊃ Aa) • (Aa ⊃ Ba) 12, Com
| 14. Ba ⊃ Aa 13, Simp
| 15. Ba 3, Simp
| 16. Aa 14, 15, MP
| 17. Aa • ~Aa 16, 9, Conj
18. ~(∀x)(Ax ≡ Bx) 10-17, IP

QED

21.
1. ~(∃x)[Rx ≡ (Tx • Ux)]
2. (∀x){(Tx ⊃ ~Ux) ⊃ [Sx ≡ (Rx ∨ Wx)]}
| 3. Rx ACP
| 4. (∀x)~[Rx ≡ (Tx • Ux)] 1, QE
| 5. ~[Rx ≡ (Tx • Ux)] 4, UI
| 6. ~{[Rx • (Tx • Ux)] ∨ [(Tx • Ux) ⊃ Rx]} 5, equiv
| 7. ~[Rx • (Tx • Ux)] • ~[(Tx • Ux) ⊃ Rx] 6, DM
| 8. ~[Rx • (Tx • Ux)] 7, simp
| 9. ~Rx ∨ ~(Tx • Ux) 8, DM
| 10. Rx ⊃ ~(Tx • Ux) 9, impl
| 11. ~(Tx • Ux) 3, 10 MP
| 12. ~Tx ∨ ~Ux 11, DM
| 13. Tx ⊃ ~Ux 12, impl
| 14. (Tx ⊃ ~Ux) ⊃ [Sx ≡ (Rx ∨ Wx)] 2, UI
| 15. Sx ≡ (Rx ∨ Wx) 13, 14 MP
| 16. [Sx ⊃ (Rx ∨ Wx)] • [(Rx ∨ Wx) ⊃ Sx] 15, equiv
| 17. [(Rx ∨ Wx) ⊃ Sx] • [Sx ⊃ (Rx ∨ Wx)] 16, com
| 18. (Rx ∨ Wx) ⊃ Sx 17, simp
| 19. Rx ∨ Wx 3, add
| 20. Sx 18, 19 MP
| 21. Sx ∨ Vx 20, add
22. Rx ⊃ (Sx ∨ Vx) 3-21 CP
23. (∀x)[Rx ⊃ (Sx ∨ Vx)] 22, UG

QED

22.

1. (∀x)[(Lx • Ix) ⊃ ~Kx]	
2. (∀x)[Mx ∨ (Jx • Nx)]	
3. (∀x)(Kx ⊃ ~Mx)	
4. (∃x)(Ix • Kx)	/ ~(∀x)(Jx ⊃ Lx)
│ 5. (∀x)(Jx ⊃ Lx)	AIP
│ 6. Ia • Ka	4, EI
│ 7. Ka • Ia	6, com
│ 8. Ka	7, simp
│ 9. Ka ⊃ ~Ma	3, UI
│ 10. ~Ma	9, 8, MP
│ 11. Ma ∨ (Ja • Na)	2, UI
│ 12. Ja • Na	11, 10, DS
│ 13. (La • Ia) ⊃ ~Ka	1, UI
│ 14. La ⊃ (Ia ⊃ ~Ka)	13, exp
│ 15. Ja	12, simp
│ 16. Ja ⊃ La	5, UI
│ 17. La	16, 15, MP
│ 18. Ia ⊃ ~Ka	14, 17, MP
│ 19. Ia	6, simp
│ 20. ~Ka	18, 19, MP
│ 21. Ka • ~Ka	8, 20, conj
22. ~(∀x)(Jx ⊃ Lx)	5-21, IP

QED

23.

1. (∀x)(Ax ≡ Dx)	
2. (∀x)[(~Bx ⊃ Cx) ⊃ Dx]	
3. (∀x)[(Ex ⊃ Bx) • (Dx ⊃ Cx)]	
4. Ax ≡ Dx	1, UI
5. (Ax ⊃ Dx) • (Dx ⊃ Ax)	4, equiv
│ 6. Ax	ACP
│ 7. Ax ⊃ Dx	5, simp
│ 8. Dx	6, 7 MP
│ 9. (Ex ⊃ Bx) • (Dx ⊃ Cx)	3, UI
│ 10. Dx ∨ Ex	8, add
│ 11. Ex ∨ Dx	10, com
│ 12. Bx ∨ Cx	9, 11 CD
13. Ax ⊃ (Bx ∨ Cx)	6-12 CP
│ 14. Bx ∨ Cx	ACP
│ 15. (~Bx ⊃ Cx) ⊃ Dx	2, UI
│ 16. ~~Bx ∨ Cx	14, DN
│ 17. ~Bx ⊃ Cx	16, impl
│ 18. Dx	15, 17 MT
│ 19. (Dx ⊃ Ax) • (Ax ⊃ Dx)	5, com
│ 20. Dx ⊃ Ax	19, simp
│ 21. Ax	18, 20 MP
22. (Bx ∨ Cx) ⊃ Ax	14-21 CP
23. [Ax ⊃ (Bx ∨ Cx)] • [(Bx ∨ Cx) ⊃ Ax]	13, 22 conj
24. Ax ≡ (Bx ∨ Cx)	23, equiv
25. (∀x)[Ax ≡ (Bx ∨ Cx)]	24, UG

QED

24.

1. (∃x)[Fx ∨ (Gx • Hx)]		
2. (∀x)[~Jx ⊃ (~Fx • ~Hx)]		
3. (∀x)(~Gx ⊃ ~Jx)		/ (∃x)(Fx ∨ Gx)
4. Fa ∨ (Ga • Ha)	1, EI	
	| 5. (∀x)~(Fx ∨ Gx)	AIP
	| 6. ~(Fa ∨ Ga)	5, UI
	| 7. ~Fa • ~Ga	6, DM
	| 8. ~Fa	7, Simp
	| 9. Ga ∨ Ha	4, 8, DS
	| 10. ~Ga • ~Fa	8, Com
	| 11. ~Ga	10, Simp
	| 12. ~Ga ⊃ ~Ja	3, UI
	| 13. ~Ja	12, 11, MP
	| 14. ~Ja ⊃ (~Fa • ~Ha)	2, UI
	| 15. ~Fa • ~Ha	14, 13, MP
	| 16. ~Ha • ~Fa	15, Com
	| 17. ~Ha	16, Simp
	| 18. Ha	9, 11, DS
	| 19. Ha • ~Ha	18, 17, Conj
20. ~(∀x)~(Fx ∨ Gx)		5-19, IP
21. (∃x)(Fx ∨ Gx)		20, QE

QED

25.

1. ~(∃x)[(Kx • Lx) • (Mx ≡ Nx)]
2. (∀x){Kx ⊃ [Ox ∨ (Px ⊃ Qx)]}
3. (∀x)[(Lx • Mx) ⊃ Px]
4. (∀x)[Nx ∨ (Kx • ~Qx)]

| 5. Lx	ACP
| 6. (∀x)~[(Kx • Lx) • (Mx ≡ Nx)]	1, QE
| 7. ~[(Kx • Lx) • (Mx ≡ Nx)]	6, UI
| 8. ~(Kx • Lx) ∨ ~(Mx ≡ Nx)	7, DM
| 9. (Kx • Lx) ⊃ ~(Mx ≡ Nx)	8, impl
| 10. Kx ⊃ [Ox ∨ (Px ⊃ Qx)]	2, UI
| 11. (Lx • Mx) ⊃ Px	3, UI
| 12. Nx ∨ (Kx • ~Qx)	4, UI
| | 13. ~(Nx ∨ Ox)	AIP
| | 14. ~Nx • ~Ox	13, DM
| | 15. ~Nx	14, simp
| | 16. Kx • ~Qx	12, 15 DS
| | 17. Kx	16, simp
| | 18. Kx • Lx	5, 17 conj
| | 19. ~(Mx ≡ Nx)	9, 18 MP
| | 20. ~[(Mx • Nx) ∨ (~Mx • ~Nx)]	19, equiv
| | 21. ~(Mx • Nx) • ~(~Mx • ~Nx)	20, DM
| | 22. ~(Mx • Nx) • ~~(Mx ∨ Nx)	21, DM
| | 23. ~(Mx • Nx) • (Mx ∨ Nx)	22, DN
| | 24. (Mx ∨ Nx) • ~(Mx • Nx)	23, com
| | 25. Mx ∨ Nx	24, simp
| | 26. Nx ∨ Mx	25, com
| | 27. Mx	15, 26 DS
| | 28. Lx • Mx	5, 27 conj
| | 29. Px	11, 28 MP
| | 30. ~Qx • Kx	16, com
| | 31. ~Qx	30, simp
| | 32. Px • ~Qx	29, 31 conj
| | 33. ~~Px • ~Qx	32, DN
| | 34. ~(~Px ∨ Qx)	33, DM
| | 35. ~(Px ⊃ Qx)	34, impl
| | 36. ~Ox • ~Nx	14, com
| | 37. ~Ox	36, simp
| | 38. ~Ox • ~(Px ⊃ Qx)	35, 37 conj
| | 39. ~[Ox ∨ (Px ⊃ Qx)]	38, DM
| | 40. ~Kx	10, 39 MT
| | 41. Kx • ~Kx	17, 40
| 42. ~~(Nx ∨ Ox)	13-41 IP
| 43. Nx ∨ Ox	42, DN
44. Lx ⊃ (Nx ∨ Ox)	5-43 CP
45. (∀x)[Lx ⊃ (Nx ∨ Ox)]	44, UG

QED

Exercises 3.6*

1. Domain: {Philosophers}

   a: Descartes; b: Hume; c: Kant; d: Godel

   Ex: x is an empiricist
   Rx: x is a rationalist
   Ox: x defends the ontological argument

2. Domain: {Animals}

   a: Debby the dolphin
   b: Harry the horse
   c: Barney the bat
   d: Sophie the bird

   Mx: x is a mammal
   Lx: x has four legs
   Wx: x has wings

3. Domain: {Cities}

   a: New York; b: Paris; c: London; d: Sydney

   Ex: x is in Europe
   Kx: koalas are native to the area around x
   Px: x is expensive

These solutions are just samples. Many alternatives are possible.

Exercises 3.7*

1. Counterexample in a 1-member universe in which:
   Aa: true
   Ba: false

2. Counterexample in a 1-member universe in which:
   Ca: false
   Da: true

3. Counterexample in a 2-member universe in which:
   Eb: false Ec: true
   Fb: true Fc: true

4. Counterexample in a 1-member universe in which:
   Ka: true
   La: true
   Ma: true
   Na: false

5. Counterexample in a 1-member universe in which:
   Gc: true
   Hc: true
   Ic: false
   Jc: true

6. Counterexample in a 2-member universe in which:
   Da: true Db: true
   Ea: true Eb: false
   Ga: false Gb: true

7. Counterexample in a 1-member universe in which:
   Pa: true
   Qa: true
   Ra: true
   Sa: false

8. Counterexample in a 4-member universe in which:
   La: true Lb: false
   Ma: false Mb: false
   Na: true Nb: true
   Oa: true Ob: false

   Lc: false Ld: false
   Mc: false Md: true
   Nc: false Nd: false
   Oc: true Od: true

9. Counterexample in a 2-member universe in which:
   Aa: true Ab: true
   Ba: true Bb: true
   Ca: true Cb: false
   Da: false Db: true

10. Counterexample in a 2-member universe in which: Pa: true Pb: false

Qa: true Qb: true
Ra: true Rb: true
Sa: false Sb: true

11. Counterexample in a 2-member universe in which:
La: true Lb: false
Ma: true Mb: true
Na: true Nb: false
Oa: true Ob: false

12. Counterexample in a 3-member universe in which:
Ia: true Ib: false Ic: true
Ja: true Jb: true Jc: false
Ka: true Kb: true Kc: false

13. Counterexample in a 2-member universe in which:
Aa: false Ab: true
Ba: false Bb: true
Ca: true Cb: false
Da: true Db: false

14. Counterexample in a 2-member universe in which:
Ra: false Rb: false
Sa: true Sb: false
Ta: true Tb: false

15. Counterexample in a 2-member universe in which:
Aa: true Ab: false
Ba: false Bb: true
Ca: false Cb: true

16. Counterexample in a 2-member universe in which:
Oa: true Ob: false
Pa: false Pb: true
Qa: false Qb: true
Ra: true Rb: false

17. Counterexample in a 3-member universe in which:
Ea: true Eb: true Ec: false
Fa: true Fb: false Fc: true
Ga: true Gb: false Gc: true

18. Counterexample in a 2-member universe in which:
Aa: true Ab: false
Ba: false Bb: true
Ca: false Cb: false
Da: false Db: true

19. Counterexample in a 2-member universe in which:
Ea: true Eb: false
Fa: false Fb: true
Ga: false Gb: true
Ha: false Hb: true

20. Counterexample in a 2-member universe in which:
Ia: false Ib: true
Ja: false Jb: true
Ka: true Kb: false
La: true Lb: false

21. Counterexample in a 4-member universe in which:
Ma: true Mb: true
Na: true Nb: true
Oa: true Ob: false
Pa: false Pb: false

Mc: false Md: false
Nc: false Nd: false
Oc: true Od: false
Pc: false Pd: true

22. Counterexample in a 2-member universe in which:
Sa: false Sb: true
Ta: true Tb: false
Ua: false Ub: true

23. Counterexample in a 2-member universe in which:
Na: true Nb: false
Oa: true Ob: false
Pa: false Pb: true
Qa: false Qb: true

24. Counterexample in a 2-member universe in which:
Ha: true Hb: false
Ia: false Ib: true
Ja: false Jb: true

25. Counterexample in a 3-member universe in which:
Ka: true Kb: false Kc: true
La: true Lb: true Lc: false
Ma: false Mb: false Mc: true

26. Counterexample in a 2-member universe in which:
Aa: true Ab: true
Ba: false Bb: true
Ca: true Cb: false

27. Counterexample in a 3-member universe in which:
Ha: true Hb: true Hc: true
Ia: false Ib: true Ic: true
Ja: true Jb: true Jc: false
Ka: false Kb: false Kc: false

28. Counterexample in a 4-member universe in which:

Pa: true	Pb: true	
Qa: true	Qb: true	
Ra: false	Rb: false	
Pc: false	Pd: false	
Qc: true	Qd: false	
Rc: true	Rd: true	

29. Counterexample in a 3-member universe in which:

Ha: true	Hb: true	Hc: false
Ia: true	Ib: false	Ic: true
Ja: true	Jb: false	Jc: true

30. Counterexample in a 3-member universe in which:

Ka: true	Kb: true	Kc: false
La: true	Lb: false	Lc: true
Ma: true	Mb: false	Mc: true
Na: true	Nb: false	Nc: false

31. Counterexample in a 3-member universe in which:

Sa: false	Sb: false	Sc: true
Ta: false	Tb: true	Tc: true
Ua: true	Ub: true	Uc: false

32. Counterexample in a 3-member universe in which:

Aa: true	Ab: true	Ac: true
Ba: true	Bb: true	Bc: false
Ca: false	Cb: false	Cc: true
Da: false	Db: false	Dc: true

33. Counterexample in a 3-member universe in which:

Fa: true	Fb: false	Fc: true
Ga: true	Gb: true	Gc: false
Ha: true	Hb: false	Hc: false

34. Counterexample in a 4-member universe in which:

Ea: false	Eb: true
Fa: false	Fb: true
Ga: false	Gb: true
Ha: false	Hb: true
Ec: true	Ed: false
Fc: false	Fd: true
Gc: true	Gd: true
Hc: true	Hd: true

35. Counterexample in a 2-member universe in which:

Pa: true	Pb: false
Qa: true	Qb: false
Ra: true	Rb: false
Sa: false	Sb: true

*Alternative counterexamples to many of these arguments are possible.

Exercises 3.8a

1. Tdc
2. ~Bej
3. Lfh
4. Tlc
5. Gba
6. Bglh
7. Ijwk
8. Mmsi
9. ~Dnos
10. Grwo

Exercises 3.8b

1. $(\exists x)(Px \bullet Sxa)$
2. $(\forall x)(Px \supset Sxa)$
3. $(\forall x)(Px \supset Sax)$
4. $\sim(\exists x)(Px \bullet Sxa)$
5. $(\forall x)[Px \supset (\exists y)(Py \bullet Sxy)]$
6. $(\exists x)[Px \bullet (\forall y)(Py \supset Sxy)]$
7. $(\forall x)(Rx \supset Lxb)$
8. $(\forall x)(Rx \supset Lbx)$
9. $(\exists x)(Rx \bullet Lxb)$
10. $\sim(\exists x)(Rx \bullet Lxb)$ or $(\forall x)(Rx \supset \sim Lxb)$
11. $(\forall x)[Rx \supset (\exists y)(My \bullet Lxy)]$
12. $(\exists x)[Rx \bullet (\exists y)(Hy \bullet Lxy)]$
13. Iocm
14. $(\exists x)(Px \bullet Ixcm)$
15. $(\exists x)[Px \bullet (\forall y)(Py \supset Ixcy)]$
16. $(\forall x)(Px \supset Sxj)$
17. $(\forall x)[Rx \supset (\exists y)(Py \bullet Hxy)]$
18. $(\forall x)[Kx \supset (\forall y)(Py \supset Lxy)]$
19. $(\forall x)[Bx \supset (\exists y)(Ny \bullet Lxy)]$
20. $(\exists x)[Cx \bullet (\forall y)(Py \supset Wxy)]$
21. $(\exists x)[Cx \bullet (\forall y)(Ly \supset Fxy)]$
22. $\sim(\exists x)[Tx \bullet (\forall y)(Cy \supset Fxy)]$
23. $\sim(\exists x)[Lx \bullet (\exists y)(Ty \bullet Fxy)]$
24. $(\exists x)[Dx \bullet (\exists y)(Py \bullet Txy)]$
25. $(\forall x)[Dx \supset (\exists y)(Py \bullet Fxy)]$
26. $\sim(\exists x)[Dx \bullet (\exists y)(Fy \bullet Cxy)]$
27. $(\forall x)[Jx \supset (\forall y)(Fy \supset Sxy)]$
28. $(\forall x)[Cx \supset (\exists y)(Sy \bullet Dxy)]$
29. $(\forall x)[Mx \supset (\exists y)(Cy \bullet Txy)]$
30. $(\exists x)[Mx \bullet (\exists y)(Sy \bullet Txy)]$
31. $(\exists x)\{Rx \bullet (\exists y)[My \bullet (\exists z)(Bz \bullet Sxyz)]\}$
32. $(\exists x)\{(Kx \bullet Px) \bullet (\exists y)[(Ey \bullet Sy) \bullet Hxy]\}$
33. $(\exists x)[(Cx \bullet Px) \bullet (\exists y)(Ay \bullet Sxy)]$
34. $(\exists x)\{Fx \bullet (\exists y)[Vy \bullet (\exists z)(Dz \bullet Rxyz)]\}$
35. $(\forall x)[(Bx \bullet Wax) \supset Rjx]$
36. $(\exists x)\{Px \bullet (\forall y)[(By \bullet Way) \supset Rxy]\}$
37. $(\forall x)(Gx \supset \sim Mxx)$
38. $(\forall x)[Sx \supset \sim(\exists y)(Px \bullet Mxy)]$
39. $(\forall x)[Px \supset (\exists y)(\exists z)(Mz \bullet Bxyz)]$
40. $\sim(\exists x)\{Mx \bullet (\forall y)[Py \supset (\exists z)Byzx]\}$

Exercises 3.8c

1. Every dog has its day.
2. What's fair for one is fair for all.
3. Rolling stones gather no moss.
4. Everything comes to those who wait.
5. God helps those who help themselves.
6. There is no place like home.
7. Every cloud has a silver lining.
8. A person is judged by the company (s)he keeps.
9. Where there's smoke, there's fire.
10. A jack of all trades is a master of none.
11. People who live in glass houses shouldn't throw stones.
12. Nothing ventured, nothing gained.

Exercises 3.9a*

1. $(\exists x)[(Ax \bullet \sim Bx) \vee (Cx \vee Dx)]$	RP1
2. $(\exists x)[Fx \supset (\exists y)Gy]$	RP9
or $(\exists y)[(\forall x)Fx \supset Gy]$	RP7
3. $(\exists x)(\exists y)[Hx \bullet (Iy \bullet Jxy)]$	RP3
4. $(\exists x)(Px \supset Ra)$	RP9
5. $(\exists x)(\forall y)[Kx \bullet (Ly \supset Mxy)]$	RP4
6. $(\forall x)[Jx \supset (\exists y)Ky]$	RP10
or $(\exists y)[(\exists x)Jx \supset Ky]$	RP7
7. $(\forall x)[(Px \bullet Qx) \supset (Ra \bullet Pa)]$	RP10
8. $(\exists x)(\exists y)[Nx \vee (Oy \bullet Pxy)]$	RP5
9. $(\forall x)[(Ex \supset Fx) \bullet (\sim Fx \equiv Gx)]$	RP2
10. $(\forall x)(\forall y)[Qx \vee (Ry \supset Sxy)]$	RP6
11. $(\forall x)(\exists y)(Dxy \supset Ex)$	RP9
12. $(\forall x)(\exists y)[Tx \supset (Uy \bullet Vxy)]$	RP7
13. $(\forall x)(\forall y)[Ax \supset (By \supset Cxy)]$	RP8
14. $(\forall x)(\forall y)[Rxy \supset (Px \bullet Qx)]$	RP10

*Alternative solutions to some of these transformations are possible.

Exercsies 3.9b*

1. (∀x)Rx • (∀x)(Tx ∨ Sx)	RP2
2. (∃x)[(Kx ≡ Lx) • (∃y)(My • Nxy)]	RP5
3. (∃x)[(Dx • Ex) ∨ (∀y)(Fy ⊃ Gxy)]	RP6
4. (∀x)[Hx ⊃ (∃y)(Iy • Jxy)]	RP3
5. (∀x)(Ox ∨ Qx) ⊃ (∃y)Py	RP9
6. (∀x)[(∃y)Axy ⊃ (Bx ∨ Cx)]	RP10
7. (∀x)[(Ax ∨ ~Cx) ⊃ (∀y)(By ⊃ Dxy)]	RP8
8. (∃x)[(Fx • Gx) • (∀y)(Hy ⊃ Exy)]	RP4
9. (∀x)[(∀y)Ixy ⊃ (Jx • Kx)]	RP9
10. (∃x)(Lx ≡ Mx) ∨ (∃x)Nx	RP1
11. (∃x)(Px • ~Qx) ⊃ (∃y)Oy	RP10
12. (∀x)[Sx ⊃ (∃y)(Ty • Rxy)]	RP7

*Alternative solutions to some of these transformations are possible.

Exercises 3.9c

1. (∃x)[Mx • (∃y)(Sy • Txy) • (∃z)(Dz • Txz)]
2. (∀x){Lx ⊃ [(∃y)(Ty • Fyx) • (∀z)(Cz ⊃ Fzx)]}
3. (∀x){(Bx ⊃ (∃y)[Ty • (∃z)(Sz • Bxyz)]}
4. (∃x){Px • (∃y){Py • (∀z)[(Bz • Wyz) ⊃ Rxz]}}
5. (∀x){(Lx • Hx) ⊃ (∃y)[(Ly • ~Hy) • Dyx]}
6. (∃x){Px • [(∃y)Oy • (∃z)(Hz • Dxyz)]}
7. (∃x){Px • ~[(∃y)Oy • (∃z)(Hz • Dxyz)]}
8. (∃x){Px • [(∃y)Oy • (∃z)(Hz • Dxzy)]}
9. (∀x){Px ⊃ ~[(∃y)Oy • (∃z)(Hz • Dxyz)]}
10. (∀x)[Sx ⊃ (∀y)(Cy ⊃ Fxy)]
11. (∃x){Rx • (∃y)[Jy • (∃z)(Ez • Sxyz)]}
12. (∀x){Rx ⊃ ~(∃x)[Jx • (∃z)(Ez • Sxyz)]}
13. (∃x){(Kx • Px) • (∃y)[Dy • (∃z)(Az • Rxyz)]}
14. (∃x){(Ex • Bx) • (∃y)[(Cy • Iy) • (∃z)(Hz • Kxyz)]}
15. (∃x)[Wx • (∃y)(Sy • Djxy)]
16. (∃x){(Px • Tx) • [(∃y)Wy • (∃z)(Sz • Dxyz)]}
17. (∃x){Cx • (∃y)[(Fy • Oxy) • Eax]}
18. (∃x){(Yx • Bx) • {(∃y)Cy • (∃z)[Fz • (Oyz • Exy)]}}
19. (∀x){Px ⊃ {(∃y)Vy • (∃z)[Fz • (Gzy • Exy)]}}
20. (∀x){(Ix • Sx) ⊃ {(∃y)By • (∃z)[Pz • (Wzy • Exy)]}}
21. (∃x){(Px • Sx) • (∃y)[(Cy • Ly) • Sxy)]}
22. (∃x){[Rx • (Cx • Sx)] • [(∃y)(Ty • Ey) • (∃z)(Bz • Bxyz)]}
23. (∃x){(Gx • Ix) • {(∃y)(Cy • Wy) • (∃z)[(Ozx • Szx) • Txyz]}}
24. (∀x){(Gx • Ix) ⊃ {(∀y)(Cy • Wy) ⊃ (∀z)[(Oxz • Szx) ⊃ ~Txyz]}}
25. (∃x){(Wx • Dx) • {(∀y)(Gy • Cy) ⊃ (∃z)[(Uz • Tz) • Pxzy]}}
26. (∀x)[(Ax • Cx) ⊃ (∃y)(Fy • Hxy)]
27. (∃x){(Mx • Fx) • (∃y)[(Ty • By) • Syx]}
28. (∀x){(Mx • Fx) ⊃ (∀y)[(Ty • By) ⊃ ~Syx]}
29. (∃x)[(Tx • Bx) • Sxx]
30. (∃x){(Tx • Bx) • (∀y)[(Py • Ty) ⊃ Sxy]}
31. (∃x){(Tx • Bx) • (∀y){[(My • Ty) • ~Syy)] ⊃ Sxy]}}

Exercises 3.9d

TBA

Exercises 3.10a

1.	1. (∀x)[(∃y)Bxy ⊃ (Ax ∨ Cx)]	
	2. (∃z)(~Az • ~Cz)	
	3. ~Aa • ~Ca	2, EI
	4. ~(Aa ∨ Ca)	3, DM
	5. (∃y)Bay ⊃ (Aa ∨ Ca)	1, UI
	6. ~(∃y)Bay	4, 5 MT
	7. (∀y)~Bay	6, QE
	8. (∃z)(∀y)~Bzy	7, EG
	QED	

2.	1. (∃x)[Qx ∨ (∃y)(Ry • Pxy)]	
	2. ~(∃x)(Sx ∨ Qx)	
	3. Qa ∨ (∃y)(Ry • Pay)	1, EI
	4. (∀x)~(Sx ∨ Qx)	2, QE
	5. ~(Sa ∨ Qa)	4, UI
	6. ~Sa • ~Qa	5, DM
	7. ~Qa • ~Sa	6, com
	8. ~Qa	7, simp
	9. (∃y)(Ry • Pay)	3, 8 DS
	10. (∃z)(∃y)(Ry • Pzy)	9, EG
	QED	

3.	1. (∀x)[(∀y)Uxy ⊃ (Tx • Vx)]	
	2. ~(∃x)Tx	
	3. (∀x)~Tx	2, QE
	4. (∀y)Uxy ⊃ (Tx • Vx)	1, UI
	5. ~Tx	3, UI
	6. ~Tx ∨ ~Vx	5, add
	7. ~(Tx • Vx)	6, DM
	8. ~(∀y)Uxy	4, 7 MT
	9. (∃y)~Uxy	8, QE
	10. ~Uxa	9, EI
	11. (∃z)~Uza	10, EG
QED		

4.	1. (∃x)[Mx • (∃y)(Ny • Lxy)]	
	2. (∀x)(∀y)(Lxy ⊃ (∃z)Oyz)	
	3. Ma • (∃y)(Ny • Lay)	1, EI
	4. (∃y)(Ny • Lay) • Ma	3, com
	5. (∃y)(Ny • Lay)	4, simp
	6. Nb • Lab	5, EI
	7. (∀y)(Lay ⊃ (∃z)Oyz)	2, UI
	8. Lab ⊃ (∃z)Obz	7, UI
	9. Lab • Nb	6, Com
	10. Lab	9, Simp
	11. (∃z)Obz	8, 10, MP
	12. Obc	11, EI
	13. (∃y)Oby	12, EG
	14. (∃x)(∃y)Oby	13, EG
QED		

5.

1. Aa • (Ba • ~Cab)	
2. (∀y)Cay ∨ (∀z)Dbz	
3. (Ba • ~Cab) • Aa	1, Com
4. Ba • ~Cab	3, Simp
5. ~Cab • Ba	4, Com
6. ~Cab	5, Simp
7. (∃y)~Cay	6, EG
8. ~(∀y)Cay	7, QE
9. (∀z)Dbz	2, 8, DS
10. (∃y)(∀z)Dyz	9, EG

QED

6.

1. (∀x)[Ex • (Fx ∨ Gx)]	
2. (∃x){Hx • (∀y)[(Fy ∨ Gy) ⊃ Ixy]}	
3. Ha • (∀y)[(Fy ∨ Gy) ⊃ Iay]	2, UI
4. (∀y)[(Fy ∨ Gy) ⊃ Iay] • Ha	3, com
5. (∀y)[(Fy ∨ Gy) ⊃ Iay]	4, simp
6. (Fy ∨ Gy) ⊃ Iay	5, UI
7. Ey • (Fy ∨ Gy)	1, UI
8. (Fy ∨ Gy) • Ey	7, com
9. Fy ∨ Gy	8, simp
10. Iay	6, 9 MP
11. (∃x)Ixy	10, EG
12. (∃y)(∃x)Ixy	11, EG

QED

7.

1. (∀x)[(Fx • Hx) ⊃ (∀y)(Gy • Ixy)]	
2. (∃x)[Jx • (∀y)(Gy ⊃ ~Ixy)]	
3. Ja • (∀y)(Gy ⊃ ~Iay)	2, EI
4. (∀y)(Gy ⊃ ~Iay) • Ja	3, com
5. (∀y)(Gy ⊃ ~Iay)	4, Simp
6. Gy ⊃ ~Iay	5, UI
7. ~Gy ∨ ~Iay	6, impl
8. ~(Gy • Iay)	7, DM
9. (∃y)~(Gy • Iay)	8, EG
10. ~(∀y)(Gy • Iay)	9, QE
11. (Fa • Ha) ⊃ (∀y)(Gy • Iay)	1, UI
12. ~(Fa • Ha)	10, 11 MT
13. (∃z)~(Fz • Hz)	12, EG
14. ~(∀z)(Fz • Hz)	13, QE

QED

8.

1. (∀x)[Ex ⊃ (∀y)(Fy • Gxy)]	
2. (∃x)(Ex • Hxb)	
3. Ea • Hab	2, EI
4. Ea	3, simp
5. Ea ⊃ (∀y)(Fy • Gay)	1, UI
6. (∀y)(Fy • Gay)	4, 5 Mp
7. Fb • Gab	6, UI
8. Gab • Fb	7, com
9. Gab	8, simp
10. Hab • Ea	3, com
11. Hab	10, simp
12. Gab • Hab	9, 11 conj
13. (∃y)(Gay • Hay)	12, EG
14. (∃x)(∃y)(Gxy • Hxy)	13, EG
QED	

9.

1. (∀x)[Ux ⊃ (∃y)(Ty • Vxy)]	
2. (∃x)Vax ⊃ (∀x)Vax	
3. Ua	/ (∃x)(∀y)Vxy
4. Ua ⊃ (∃y)(Ty • Vay)	1, UI
5. (∃y)(Ty • Vay)	4, 3, MP
6. Tb • Vab	5, EI
7. Vab • Tb	6, com
8. Vab	7, simp
9. (∃x)Vax	8, EG
10. (∀x)Vax	2, 9, MP
11. Vay	10, UI
12. (∀y)Vay	11, UG
13. (∃x)(∀y)Vxy	12, EG

QED

10.

1. (∀x)(∀y)[Ax ⊃ (Dy ⊃ Byx)]	
2. (∃x)(∀y)[Dx • (Bxy ⊃ Cy)]	
| 3. (∀x)Ax	ACP
| 4. (∀y)[Da • (Bay ⊃ Cy)]	2, EI
| 5. Da • (Baa ⊃ Ca)	4, UI
| 6. Aa	3, UI
| 7. (∀y)[Aa ⊃ (Dy ⊃ Bya)]	1, UI
| 8. Aa ⊃ (Da ⊃ Baa)	7, UI
| 9. Da ⊃ Baa	8, 6, MP
| 10. Da	5, simp
| 11. (Baa ⊃ Ca) • Da	5, com
| 12. Baa ⊃ Ca	11, simp
| 13. Da ⊃ Ca	9, 12 HS
| 14. Ca	13, 10 MP
| 15. (∃y)Cy	14, EG
16. (∀x)Ax ⊃ (∃y)Cy	3-15 CP

QED

11.

1. (∀x)[Ax ⊃ (∃y)(Cy • Dxy)}		
2. (∀x)(∀y)(Dxy ⊃ By)		
	| 3. (∀x)Ax	ACP
	| 4. Ax	3, UI
	| 5. Ax ⊃ (∃y)(Cy • Dxy)	1, UI
	| 6. (∃y)(Cy • Dxy)	4, 5 MP
	| 7. Ca • Dxa	6, EI
	| 8. Dxa • Ca	7, com
	| 9. Dxa	8, simp
	| 10. (∀y)(Dxy ⊃ By)	2, UI
	| 11. Dxa ⊃ Ba	10, UI
	| 12. Ba	9, 11 MP
	| 13. Ca	7, simp
	| 14. Ba • Ca	12, 13 conj
	| 15. (∃y)(By • Cy)	14, EG
16. (∀x)Ax ⊃ (∃y)(By • Cy)		3-15 CP
QED		

12.

1. (∃x){Px • (∀y)[Oy ⊃ (∀z)(Rz ⊃ Qxyz)]}	
2. (∀x)[Px ≡ (Ox • Rx)]	
3. Pa • (∀y)[Oy ⊃ (∀z)(Rz ⊃ Qayz)]	1, EI
4. Pa ≡ (Oa • Ra)	2, UI
5. Pa	3, simp
6. [Pa ⊃ (Oa • Ra)] • [(Oa • Ra) ⊃ Pa]	4, equiv
7. Pa ⊃ (Oa • Ra)	6, simp
8. Oa • Ra	5, 7 MP
9. Oa	8, simp
10. Ra • Oa	8, com
11. Ra	10, simp
12. (∀y)[Oy ⊃ (∀z)(Rz ⊃ Qayz)] • Pa	3, com
13. (∀y)[Oy ⊃ (∀z)(Rz ⊃ Qayz)]	12, simp
14. Oa ⊃ (∀z)(Rz ⊃Qaaz)	13, UI
15. (∀z)(Rz ⊃ Qaaz)	9, 14 MP
16. Ra ⊃ Qaaa	15, UI
17. Qaaa	11, 16 MP
18. (∃x)Qxxx	17, EG
QED	

13.	1. $(\forall x)(Mx \supset {\sim}Ox) \supset (\exists y)Ny$	
	2. $(\forall y)[Ny \supset (\exists z)(Pz \bullet Qyz)]$	
	3. ${\sim}(\exists x)(Mx \bullet Ox)$	/ $(\exists x)[Nx \bullet (\exists y)Qxy]$
	4. $(\forall x){\sim}(Mx \bullet Ox)$	3, QE
	5. $(\forall x)({\sim}Mx \vee {\sim}Ox)$	4, DM
	6. $(\forall x)(Mx \supset {\sim}Ox)$	5, Impl
	7. $(\exists y)Ny$	1, 6, MP
	8. Na	7, EI
	9. $Na \supset (\exists z)(Pz \bullet Qaz)$	2, UI
	10. $(\exists z)(Pz \bullet Qaz)$	9, 8, MP
	11. $Pb \bullet Qab$	10, EI
	12. $Qab \bullet Pb$	11, com
	13. Qab	11, simp
	14. $(\exists y)Qay$	13, EG
	15. $Na \bullet (\exists y)Qay$	8, 14, Conj
	16. $(\exists x)[Nx \bullet (\exists y)Qay]$	15, EG
QED		

14.	1. $(\forall x)(\forall y)[Kx \supset (My \supset Lxy)]$	
	2. $(\exists x)(\exists y)[Mx \bullet (Ky \bullet Nxy)]$	
	3. $(\exists y)[Ma \bullet (Ky \bullet Nay)]$	2, EI
	4. $Ma \bullet (Kb \bullet Nab)$	3, EI
	5. $(Ma \bullet Kb) \bullet Nab$	4, assoc
	6. $Ma \bullet Kb$	5, simp
	7. Ma	6, simp
	8. $Kb \bullet Ma$	6, com
	9. Kb	8, simp
	10. $(\forall y)[Kb \supset (My \supset Lby)]$	1, UI
	11. $Kb \supset (Ma \supset Lba)$	10, UI
	12. $Ma \supset Lba$	9, 11 MP
	13. Lba	7, 12 MP
	14. $Nab \bullet (Ma \bullet Kb)$	5, com
	15. Nab	14, simp
	16. $Lba \bullet Nab$	13, 15 conj
	17. $(\exists x)(Lxa \bullet Nax)$	16, EG
	18. $(\exists y)(\exists x)(Lxy \bullet Nyx)$	17, EG
	QED	

15.	1. (∀x)[Rx ⊃ (∀y)(Ty ⊃ Uxy)]	
	2. (∀y)[(∀x)(Uxy ⊃ Sy)]	
	│3. Rx • Tx	ACP
	│4. Rx	3, simp
	│5. Tx • Rx	3, com
	│6. Tx	5, simp
	│7. (∃x)Tx	6, EG
	│8. Tb	6, EI
	│9. Rx ⊃ (∀y)(Ty ⊃ Uxy)	1, UI
	│10. (∀y)(Ty ⊃ Uxy)	4, 9 MP
	│11. Tb ⊃ Uxb	10, UI
	│12. (∀x)(Uxb ⊃ Sb)	2, UI
	│13. Uxb ⊃ Sb	12, UI
	│14. Tb ⊃ Sb	11, 13 HS
	│15. Sb	8, 14 MP
	│16. (∃y)Sy	15, EG
	17. (Rx • Tx) ⊃ (∃y)Sy	3-15 CP
	18. (∀x)[(Rx • Tx) ⊃ (∃y)Sy]	17, UG
	QED	

16.	1. (∃x)(∀y)[(Fx • Dx) ∨ (Ey ⊃ Gxy)]	
	2. (∀x)[(∃y)Gxy ⊃ (∃z)Hxz]	
	3. ~(∃x)Fx • (∀z)Ez	
	4. (∀y)[(Fa • Da) ∨ (Ey ⊃ Gay)]	1, EI
	5. (Fa • Da) ∨ (Ez ⊃ Gaz)	4, UI
	6. ~(∃x)Fx	3, simp
	7. (∀x)~Fx	6, QE
	8. ~Fa	7, UI
	9. ~Fa ∨ ~Ha	8, add
	10. ~(Fa • Ha)	9, DM
	11. Ez ⊃ Gaz	5, 10 MP
	12. (∀z)Ez • ~(∃x)Fx	3, com
	13. (∀z)Ez	12, simp
	14. Ez	13, UI
	15. Gaz	11, 14 MP
	16. (∃y)Gay	15, EG
	17. (∃y)Gay ⊃ (∃z)Haz	2, UI
	18. (∃z)Haz	16, 17 MP
	19. (∃y)(∃z)Hyz	18, EG
	QED	

17.

1.	(∀x)(Kx ≡ Lx) • (∀x)Jx	
2.	(∀x)[Jx ⊃ (∃y)(~Ky • Mxy)]	
3.	(∀x)Jx • (∀x)(Kx ≡ Lx)	1, com
4.	(∀x)Jx	3, simp
5.	Jx	4, UI
6.	Jx ⊃ (∃y)(~Ky • Mxy)	2, UI
7.	(∃y)(~Ky • Mxy)	5, 6 MP
8.	~Ka • Mxa	7, EI
9.	(∀x)(Kx ≡ Lx)	1, simp
10.	Ka ≡ La	9, UI
11.	(Ka ⊃ La) • (La ⊃ Ka)	10, equiv
12.	(La ⊃ Ka) • (Ka ⊃ La)	11, com
13.	La ⊃ Ka	12, simp
14.	~Ka	8, simp
15.	~La	13, 14 MT
16.	Mxa • ~Ka	8, com
17.	Mxa	16, simp
18.	~La • Mxa	15, 17 conj
19.	(∃x)(~Lx • Mxx)	18, EG

QED

18.

1.	(∀x)[Kx ⊃ (∃y)(Jy • Ixy)]	
2.	(∀x)(∀y)(Ixy ⊃ Lx)	
3.	~(~Kx ∨ Lx)	AIP
4.	~~Kx • ~Lx	3, DM
5.	Kx • ~Lx	4, DN
6.	Kx	5, simp
7.	Kx ⊃ (∃y)(Jy • Ixy)	1, UI
8.	(∃y)(Jy • Ixy)	6, 7 MP
9.	Ja • Ixa	8, EI
10.	Ixa • Ja	9, com
11.	Ixa	10, simp
12.	(∀y)(Ixy ⊃ Lx)	3, UI
13.	Ixa ⊃ Lx	12, UI
14.	Lx	11, 13 MP
15.	~Lx • Kx	5, com
16.	~Lx	15, simp
17.	Lx • ~Lx	14, 16 conj
18.	~~(~Kx ∨ Lx)	3-17 IP
19.	~Kx ∨ Lx	18, DN
20.	(∀x)(~Kx ∨ Lx)	19, UG

QED

19.

1. (∀x)[(Ox ⊃ Nx) ⊃ (∀y)(Qy • ~Rxy)]	
2. (∀y)(∀x)(Pxy ⊃ Rxy)	
│3. Nx ∨ ~Ox	ACP
│4. ~Ox ∨ Nx	3, com
│5. Ox ⊃ Nx	4, impl
│6. (Ox ⊃ Nx) ⊃ (∀y)(Qy • ~Rxy)	1, UI
│7. (∀y)(Qy • ~Rxy)	5, 6 MP
│8. Qy • ~Rxy	7, UI
│9. ~Rxy • Qy	8, com
│10. ~Rxy	9, simp
│11. (∀x)(Pxy ⊃ Rxy)	2, UI
│12. Pxy ⊃ Rxy	11, UI
│13. ~Pxy	10, 12 MT
│14. Qy	8, simp
│15. Qy • ~Pxy	13, 14 conj
│16. ~~Qy • ~Pxy	15, DN
│17. ~(~Qy ∨ Pxy)	16, DM
│18. ~(Qy ⊃ Pxy)	17, impl
│19. (∀y)~(Qy ⊃ Pxy)	18, UG
20. (Nx ∨ ~Ox) ⊃ (∀y)~(Qy ⊃ Pxy)	3-19 CP
21. (∀x)[(Nx ∨ ~Ox) ⊃ (∀y)~(Qy ⊃ Pxy)]	20, UG

QED

20.

1. (∀x)[(Fx ≡ Hx)	
2. (∀x)(Hx ⊃ ~Ix)	
3. (∃x)[Fx • (∃y)(Iy • ~Gxy)]	/ (∃x)[(Fx • ~Ix) • (∃y)(Iy • ~Gxy)]
4. Fa • (∃y)(Iy • ~Gay)	3, EI
5. Fa	4, Simp
6. Fa ≡ Ha	1, UI
7. (Fa ⊃ Ha) • (Ha ⊃ Fa)	6, Equiv
8. Fa ⊃ Ha	7, Simp
9. Ha	8, 5, MP
10. Ha ⊃ ~Ix	2, UI
11. ~Ia	10, 9, MP
12. Fa • ~Ia	5, 11, Conj
13. (∃y)(Iy • ~Gay) • Fa	4, Com
14. (∃y)(Iy • ~Gay)	13, Simp
15. (Fa • ~Ia) • (∃y)(Iy • ~Gay)	12, 14, Conj
16. (∃x)[(Fx • ~Ix) • (∃y)(Iy • ~Gxy)]	15, EG

QED

21.

1. (∀x){Ax ⊃ (∃y)[By • (∀z)(~Cz • Dzxy)]}	
2. ~(∀x)(Ax ⊃ Cx)	/ (∃x)(∃y)Dxxy
3. (∃x)~(Ax ⊃ Cx)	2, QE
4. (∃x)~(~Ax ∨ Cx)	3, Impl
5. (∃x)(~~Ax • ~Cx)	4, DM
6. (∃x)(Ax • ~Cx)	5, DN
7. Aa • ~Ca	6, EI
8. Aa	7, Simp
9. Aa ⊃ (∃y)[By • (∀z)(~Cz • Dzay)]}	1, UI
10. (∃y)[By • (∀z)(~Cz • Dzay)]}	9, 8, MP
11. Bb • (∀z)(~Cz • Dzab)	10, EI
12. (∀z)(~Cz • Dzab) • Bb	11, Com
13. (∀z)(~Cz • Dzab)	12, Simp
14. ~Ca • Daab	13, UI
15. Daab • ~Ca	14, Com
16. Daab	15, Simp
17. (∃y)Daay	16, EG
18. (∃x)(∃y)Dxxy	17, Eg

QED

22.

1. (∀x)[(Bx ⊃ Ax) ⊃ (∃y)(Cy • Dxy)]	
2. (∀x)[(∀y)~Dxy ∨ Ex]	
3. (∃x)Ex ⊃ ~(∃x)Cx	
4. (∃x)~Bx	AIP
5. ~Ba	4, EI
7. ~Ba ∨ Aa	4, add
8. Ba ⊃ Aa	5, impl
9. (Ba ⊃ Aa) ⊃ (∃y)(Cy • Day)	1, UI
10. (∃y)(Cy • Day)	6, 7 MP
11. Cb • Dab	8, EI
12. Cb	9, simp
13. (∃x)Cx	12, EG
14. ~~(∃x)Cx	13, DN
15. ~(∃x)Ex	3, 14, MT
16. (∀x)~Ea	15, QE
17. ~Ea	16, UI
18. (∀y)~Day ∨ Ea	2, UI
19. Ea ∨ (∀y)~Day	18, Com
20. (∀y)~Day	19, 17, DS
21. ~Dab	20, UI
22. Dab • Cb	11, Com
23. Dab	22, Simp
24. Dab • ~Dab	23, 21, Conj
25. ~(∃x)~Bx	4-24, IP
26. (∀x)Bx	25, QE

QED

23.

1. (∀x){(Tx ⊃ ~Sx) ⊃ (∃y)[Uy ∨ (∀z)(Vz ⊃ Wxyz)]}	
2. ~(∃x)(Tx ≡ Sx)	
3. ~(∃x)(Vx ⊃ Ux)	
4. (Tx ⊃ ~Sx) ⊃ (∃y)[Uy ∨ (∀z)(Vz ⊃ Wxyz)]	1, UI
5. (∀x)~(Tx ≡ Sx)	2, QE
6. ~(Tx ≡ Sx)	5, UI
7. ~[(Tx • Sx) ∨ (~Tx • ~Sx)]	6, equiv
8. ~(Tx • Sx) • ~(~Tx • ~Sx)	7, DM
9. ~(Tx • Sx)	8, simp
10. ~Tx ∨ ~Sx	9, DM
11. Tx ⊃ ~Sx	10, impl
12. (∃y)[Uy ∨ (∀z)(Vz ⊃ Wxyz)]	4, 11 MP
13. Ub ∨ (∀z)(Vz ⊃ Wxbz)	12, EI
14. (∀x)~(Vx ⊃ Ux)	3, QE
15. ~(Vb ⊃ Ub)	14, UI
16. ~(~Vb ∨ Ub)	15, impl
17. ~~Vb • ~Ub	16, DM
18. Vb • ~Ub	17, DN
19. Vb	18, simp
20. ~Ub • Vb	18, com
21. ~Ub	20, simp
22. (∀z)(Vz ⊃ Wxbz)	13, 21 DS
23. Vb ⊃ Wxbb	22, UI
24. Wxbb	19, 23 MP
25. (∃y)Wxyy	24, EG
26. (∃x)(∃y)Wxyy	25, EG

QED

24.

1. (∀x)[Fx ⊃ (∃y)(Hy • Gxy)]	
2. (∀x)[Hx ⊃ (∃y)(Ey • Gxy)]	
3. (∀x)[Ex ⊃ (∀y)Fy]	/ (∀x)Fx ≡ (∃x)Ex
│ 4. (∀x)Fx	ACP
│ 5. Fx	4, UI
│ 6. Fx ⊃ (∃y)(Hy • Gxy)	1, UI
│ 7. (∃y)(Hy • Gxy)	6, 5, MP
│ 8. Ha • Gxa	7, EI
│ 9. Ha	8, simp
│ 10. Ha ⊃ (∃y)(Ey • Gay)	2, UI
│ 11. (∃y)(Ey • Gay)	10, 9, MP
│ 12. Eb • Gab	11, EI
│ 13. Eb	12, simp
│ 14. (∃x)Ex	13, EG
15. (∀x)Fx ⊃ (∃x)Ex	5-14 CP
│ 16. (∃x)Ex	ACP
│ 17. Ec	16, EI
│ 18. Ec ⊃ (∀y)Fy	3, UI
│ 19. (∀y)Fy	18, 17, MP
│ 20. Fx	19, UI
│ 21. (∀x)Fx	20, UG
22. (∃x)Ex ⊃ (∀x)Fx	16-21, CP
23. [(∀x)Fx ⊃ (∃x)Ex] • [(∃x)Ex ⊃ (∀x)Fx]	15, 22, Conj
24. (∀x)Fx ≡ (∃x)Ex	23, equiv

QED

25.

1. (∀x){Jx ⊃ (∀y)[My ⊃ (∀z)(Lz ⊃ Kxyz)]}
2. (∃x)(∃y)[Mx • (Jy • Nxy)]

3. ~(∀x)(Lx ⊃ Ox) / (∃x){Mx • (∃y)[Nxy • (∃z)(~Oz • Kyxz)]})
4. (∃x)~(Lx ⊃ Ox) 3, QE
5. ~(La ⊃ Oa) 4, EI
6. ~(~La ∨ Oa) 5, impl
7. ~~La • ~Oa 6, DM
8. ~~La 7, simp
9. La 8, DN
10. (∃y)[Mb • (Jy • Nby)] 2, EI
11. Mb • (Jc • Nbc) 10, EI
12. Mb 11, simp
13. (Jc • Nbc) • Mb 11, com
14. Jc • Nbc 13, simp
15. Jc 14, simp
16. Nbc • Jc 14, com
17. Nbc 16, simp
18. Jc ⊃ (∀y)[My ⊃ (∀z)(Lz ⊃ Kcyz)] 1, UI
19. (∀y)[My ⊃ (∀z)(Lz ⊃ Kcyz)] 15, 18 MP
20. Mb ⊃ (∀z)(Lz ⊃ Kcbz) 19, UI
21. (∀z)(Lz ⊃ Kcbz) 12, 20 MP
22. La ⊃ Kcba 21, UI
23. Kcba 9, 22 MP
24. ~Oa • ~~La 7, Com
25. ~Oa 24, Simp
26. ~Oa • Kcba 25, 23, Conj
27. (∃z)(~Oz • Kcbz) 26, EG
28. Nbc • (∃z)(~Oz • Kcbz) 17, 27, Conj
29. (∃y)[Nby • (∃z)(~Oz • Kybz)] 28, EG
30. Mb • (∃y)[Nby • (∃z)(~Oz • Kybz)] 12, 29, Conj
31. (∃x){Mx • (∃y)[Nxy • (∃z)(~Oz • Kyxz)]}) 20, EG

QED

26.
1. (∀x)[Tx ⊃ (∀y)(Vy ⊃ Uxy)]
2. ~(∃x)(Tx • Sx)
3. Ta • Vb / (∃x)[~Sx • (∃y)Uxy]
4. Ta ⊃ (∀y)(Vy ⊃ Uay) 1, UI
5. Ta 3, Simp
6. (∀y)(Vy ⊃ Uay) 4, 5, MP
7. Vb • Ta 3, Com
8. Vb 7, Simp
9. Vb ⊃ Uab 6, UI
10. Uab 9, 8, MP
11. (∀x)~(Tx • Sx) 2, QE
12. ~(Ta • Sa) 11, UI
13. ~Ta ∨ ~Sa 12, DM
14. ~~Ta 5, DN
15. ~Sa 13, 14, DS
16. (∃y)Uay 10, EG
17. ~Sa • (∃y)Uay 15, 16, Conj
18. (∃x)[~Sx • (∃y)Uxy] 17, EG

QED

Exercises 3.10b

1.	1. (∃x)[Px • (∃y)(Oy •Ixy)]	
	2. (∃x)[Fx • (∀y)(Py ⊃ ~Sxy)]	/ (∃x)[Fx • (∃y)~Sxy]
	3. Fa • (∀y)(Py ⊃ ~Say)	2, EI
	4. Fa	3, Simp
	5. (∀y)(Py ⊃ ~Say) • Fa	3, Com
	6. (∀y)(Py ⊃ ~Say)	5, Simp
	7. Pb • (∃y)(Oy • Iby)	1, EI
	8. Pb	7, Simp
	9. Pb ⊃ ~Sab	6, UI
	10. ~Sab	9, 8, MP
	11. (∃y)~Say	10, EG
	12. Fa • (∃y)~Say	4, 11, Conj
	13. (∃x)[Fx • (∃y)~Sxy]	12, EG

QED

2.	1. (∃x)[Bx • (∃y)(Gy • Sxy)]	
	2. (∀y)(Gy ⊃ ~Cy)	/ ~(∀x)Ex
	3. Ba • (∃y)(Gy • Say)	1, EI
	4. (∃y)(Gy • Say) • Ba	3, com
	5. (∃y)(Gy • Say)	4, simp
	6. Gb • Sab	5, EI
	7. Gb ⊃ ~Cb	2, UI
	8. Gb	6, simp
	9. ~Cb	7, 8 MP
	10. (∃x)~Cx	9, EG
	11. ~(∀x)Cx	10, QE

QED

3.	1. (∀x){[Px • (∃y)(My • Txy)] ⊃ Ix}	
	2. Pr • Trc	
	3. Mc	/ Ir
	4. [Pr • (∃y)(My • Try)] ⊃ Ir	1, UI
	5. Pr	2, Simp
	6. Trc • Pr	2, Com
	7. Trc	6, Simp
	8. Mc • Trc	3, 7, Conj
	9. (∃y)(My • Try)	8, EG
	10. Pr • (∃y)(My • Try)	5, 9, Conj
	11. Ir	4, 10, MP

QED

4.	1. (∀x)[Cx ⊃ (∀y)(Dy ⊃ Lxy)]	
	2. ~(∀x)Lxb • (∀x)Cx	/ ~(∀y)Dy
	3. ~(∀x)Lxb	2, simp
	4. (∃x)~Lxb	3, QE
	5. ~Lab	4, EI
	6. (∀x)Cx • ~(∀x)Lxb	2, com
	7. (∀x)Cx	6, simp
	8. Ca	7, UI
	9. Ca ⊃ (∀y)(Dy ⊃ Lay)	1, UI
	10. (∀y)(Dy ⊃ Lay)	8,9 MP
	11. Db ⊃ Lab	10, UI
	12. ~Db	5, 11 MT
	13. (∃x)~Dx	12, EG
	14. ~(∀x)Dx	13, QE
	QED	

5.	1. (∀x)[Cx ⊃ (∃y)(Ty • Fxy)]	
	2. (∀y)(Sy ≡ Ty)	/ (∃x)Cx ⊃ (∃x)Sx
	│3. (∃x)Cx	ACP
	│4. Ca	3, EI
	│5. Ca ⊃ (∃y)(Ty • Fay)	1, UI
	│6. (∃y)(Ty • Fay)	4, 5 MP
	│7. Tb • FaB	6, EI
	│8. Tb	7, simp
	│9. Sb ≡ Tb	2, UI
	│10. (Sb ⊃ Tb) • (Tb ⊃ Sb)	9, equiv
	│11. (Tb ⊃ Sb) • (Sb ⊃ Tb)	10, com
	│12. Tb ⊃ Sb	11, simp
	│13. Sb	8, 12 MP
	│14. (∃x)Sx	13, EG
	15. (∃x)Cx ⊃ (∃x)Sx	3-14 CP
	QED	

6.	1. (∃x)[Bx • (∃y)(Sy • Baxy)]	
	2. (∀x)(Bx ⊃ Fx)	
	3. Lac	/ (∃x){Lxc • (∃y)[Fy • (∃z)(Sz • Bxyz)]}
	4. Bb • (∃y)(Sy • Baby)	1, EI
	5. Bb	4, Simp
	6. Bb ⊃ Fb	2, UI
	7. Fb	6, 5, MP
	8. (∃y)(Sy • Baby) • Bb	4, Com
	9. (∃y)(Sy • Baby)	8, Simp
	10. Sd • Babd	9, EI
	11. (∃z)(Sz • Babz)	10, EG
	12. Fb • (∃z)(Sz • Babz)	7, 12, Conj
	13. (∃y)[Fy • (∃z)(Sz • Bayz)]	12, EG
	14. Lac • (∃y)[Fy • (∃z)(Sz • Bayz)]	3, 13, Conj
	15. (∃x){Lxc • (∃y)[Fy • (∃z)(Sz • Bxyz)]}	14, EG

QED

7.

	1. (∀x)[Px ⊃ (∃y)(My • Rxy)]	
	2. (∃x)~Px ∨ (∀x)~Mx	/ ~(∀x)Px
|	3. (∀x)Px	AIP
|	4. ~(∃x)~Px	3, QE
|	5. (∀x)~Mx	2, 4, DS
|	6. Px ⊃ (∃y)(My • Rxy)	1, UI
|	7. Px	3, UI
|	8. (∃y)(My • Rxy)	6, 7, MP
|	9. Ma • Ray	8, EI
|	10. Ma	9, Simp
|	11. ~Ma	5, UI
|	12. Ma • ~Ma	10, 11, Conj
	13. ~(∀x)Px	3-12, IP

QED

8.

1. (∃x){Sx • (∃y){[By • (∃z)(Pz • Wzy)] • Rxy}}	
2. (∀x){[Bx • (∃y)(Py • Wyx)] ⊃ Wx}	/ (∃x){Px • (∃y)[(By • Wy) • Wxy]}
3. Sa • (∃y){[By • (∃z)(Pz • Wzy)] • Ray}	1, UI
4. (∃y){[By • (∃z)(Pz • Wzy)] • Ray} • Sa	3, Com
5. (∃y){[By • (∃z)(Pz • Wzy)] • Ray}	4, Simp
6. [Bb • (∃z)(Pz • Wzb)] • Rab	5, EI
7. Bb • (∃z)(Pz • Wzb)	6, Simp
8. Bb	7, Simp
9. (∃z)(Pz • Wzb) • Bb	7, Com
10. (∃z)(Pz • Wzb)	9, Simp
11. Pc • Wcb	10, EI
12. (∃y)(Py • Wyb)	11, EG
13. Bb • (∃y)(Py • Wyb)	8, 12, Conj
14. [Bb • (∃y)(Py • Wyb)] ⊃ Wb	2, UI
15. Wb	14, 13, MP
16. Pc	11, Simp
17. Wcb • Pc	11, Com
18. Wcb	17, Simp
19. Bb • Wb	8, 15, Conj
20. (Bb • Wb) • Wcb	19, 18, Conj
21. (∃y)[(By • Wy) • Wcy]	20, EG
22. Pc • (∃y)[(By • Wy) • Wcy]	16, 21, Conj
23. (∃x){Px • (∃y)[(By • Wy) • Wxy]}	22, EG

QED

9.	1. (∀x)[(Sx ∨ Rx) ⊃ Px]	
	2. (∃x)[Sx • (∀y)(Ry ⊃ Gxy)]	
	3. (∃x)(Rx • Rrxe)	/ (∃x)[Px • (∃y)(Ry • Gxy)]
	4. Sa • (∀y)(Ry ⊃ Gay)	2, EI
	5. Sa	4, Simp
	6. Sa ∨ Ra	5, Add
	7. (Sa ∨ Ra) ⊃ Pa	1, UI
	8. Pa	7, 6, MP
	9. (∀y)(Ry ⊃ Gay) • Sa	4, Com
	10. (∀y)(Ry ⊃ Gay)	9, Simp
	11. Rb • Rrbe	3, EI
	12. Rb	11, Simp
	13. Rb ⊃ Gab	10, UI
	14. Gab	13, 12, MP
	15. Rb • Gab	12, 14, Conj
	16. (∃y)(Ry • Gay)	15, EG
	17. Pa • (∃y)(Ry • Gay)	8, 16, Conj
	18. (∃x)[Px • (∃y)(Ry • Gxy)]	17, EG
QED		

10.	1. (∀x)[(Ex ≡ Qx) ∨ Tx]	
	2. (∃x)(Mx • Px)	
	3. (∀x)(Mx ⊃ Ex)	
	4. (∀x)~Qx	/ (∃x)(Tx • Px)
	5. Ma • Pa	2, EI
	6. Ma	2, Simp
	7. Ma ⊃ Ea	3, UI
	8. Ea	7, 6, MP
	9. ~Qa	4, UI
		10. ~(∃x)(Tx • Px)	AIP
		11. (∀x)~(Tx • Px)	10, QE
		12. ~(Ta • Pa)	11, UI
		13. ~Ta ∨ ~Pa	12, DM
		14. ~Pa ∨ ~Ta	13, Com
		15. Pa • Ma	5, Com
		16. Pa	15, Simp
		17. ~~Pa	16, DN
		18. ~Ta	14, 17, DS
		19. (Ea ≡ Qa) ∨ Ta	1, UI
		20. Ta ∨ (Ea ≡ Qa)	19, Com
		21. Ea ≡ Qa	20, 18, DS
		22. (Ea ⊃ Qa) • (Qa ⊃ Ea)	21, Equiv
		23. Ea ⊃ Qa	22, Simp
		24. Qa	23, 8, MP
		25. Qa • ~Qa	24, 9, Conj
	26. ~~(∃x)(Tx • Px)	10-25, IP
	27. (∃x)(Tx • Px)	26, DN
QED		

Exercises 3.10c

1.

	1. ~(∀y)[Fy ⊃ (∃x)Fx]	AIP
	2. (∃y)~[Fy ⊃ (∃x)Fx]	1, QE
	3. (∃y)~[~Fy ∨ (∃x)Fx]	2, Impl
	4. (∃y)[~~Fy • ~(∃x)Fx]	3, DM
	5. (∃y)[Fy • ~(∃x)Fx]	4, DM
	6. Fa • ~(∃x)Fx	5, EI
	7. Fa	6, Simp
	8. ~(∃x)Fx • Fa	6, Com
	9. ~(∃x)Fx	8, Simp
	10. (∃x)Fx	7, EG
	11. (∃x)Fx • ~(∃x)Fx	10, 9, Conj
12. ~~(∀y)[Fy ⊃ (∃x)Fx]	1-11 IP
13. (∀y)[Fy ⊃ (∃x)Fx]	12, DN

QED

2.

	1. ~(∃y)[Fy ⊃ (∀x)Fx]	AIP
	2. (∀y)~[Fy ⊃ (∀x)Fx]	1, QE
	3. (∀y)~[~Fy ∨ (∀x)Fx]	2, Impl
	4. (∀y)[~~Fy • ~(∀x)Fx]	3, DM
	5. (∀y)[Fy • ~(∀x)Fx]	4, DN
	6. Fy • ~(∀x)Fx	5, UI
	7. Fy	6, Simp
	8. (∀x)Fx	7, UG
	9. ~(∀x)Fx • Fy	6, Com
	10. ~(∀x)Fx	9, Simp
	11. (∀x)Fx • ~(∀x)Fx	8, 10, Conj
12. ~~(∃y)[Fy ⊃ (∀x)Fx]	1-11, IP
13. (∃y)[Fy ⊃ (∀x)Fx]	12, DN

QED

3.

	1. ~(∃y)[(∃x)Fx ⊃ Fy]	AIP
	2. (∀y)~[(∃x)Fx ⊃ Fy]	1, QE
	3. (∀y)~[~(∃x)Fx ∨ Fy]	2, Impl
	4. (∀y)[~~(∃x)Fx • ~Fy]	3, DM
	5. (∀y)[(∃x)Fx • ~Fy]	4, DN
	6. (∃x)Fx • ~Fy	5, UI
	7. ~Fy • (∃x)Fx	6, Com
	8. ~Fy	7, Simp
	9. (∀x)~Fx	8, UG
	10. ~(∃x)Fx	9, QE
	11. (∃x)Fx	6, Simp
	12. (∃x)Fx • ~(∃x)Fx	11, 10, Conj
13. ~~(∃y)[(∃x)Fx ⊃ Fy]	1-12, IP
14. (∃y)[(∃x)Fx ⊃ Fy]	13, DN

QED

4.
| 1. (∃x)(∀y)Cxy ACP
| | 2. ~(∀y)(∃x)Cxy AIP
| | 3. (∃y)~(∃x)Cxy 2, QE
| | 4. (∃y)(∀x)~Cxy 3, QE
| | 5. (∀y)Cay 1, EI
| | 6. (∀x)~Cxb 4, EI
| | 7. ~Cab 6, UI
| | 8. Cab 5, UI
| | 9. ~Cab • Cab 7, 8 conj
| 10. ~~(∀y)(∃x)Cxy 2-9 IP
| 11. (∀y)(∃x)Cxy 10, DN
12. (∃x)(∀y)Cxy ⊃ (∀y)(∃x)Cxy 1-11, CP
QED

5.
| 1. (∀x)(∃y)Hxy ACP
| | 2. ~(∃x)(∃y)Hxy AIP
| | 3. (∀x)~(∃y)Hxy 2, QE
| | 4. (∀x)(∀y)~Hxy 3, QE
| | 5. (∃y)Hxy 1, UI
| | 6. Hxa 5, EI
| | 7. (∀y)~Hxy 4, UI
| | 8. ~Hxa 7, UI
| | 9. Hxa • ~Hxa 6, 8 conj
| 10. ~~(∃x)(∃y)Hxy 2-9 IP
| 11. (∃x)(∃y)Hxy 10, DN
12. (∀x)(∃y)Hxy ⊃ (∃x)(∃y)Hxy 1-11, CP
QED

6.
| 1. ~Fa ACP
| | 2. ~[(∀x)Fx ⊃ Ga] AIP
| | 3. ~[~(∀x)Fx ∨ Ga] 2, impl
| | 4. ~~(∀x)Fx • ~Ga 3, DM
| | 5. (∀x)Fx • ~Ga 4, DN
| | 6. (∀x)Fx 5, simp
| | 7. Fa 6, UI
| | 8. ~Fa • Fa 1, 7 conj
| 9. ~~[(∀x)Fx ⊃ Ga] 2-8 IP
| 10. (∀x)Fx ⊃ Ga 9, DN
11. ~Fa ⊃ [(∀x)Fx ⊃ Ga] 1-10, CP
12. ~~Fa ∨ [(∀x)Fx ⊃ Ga] 11, impl
13. Fa ∨ [(∀x)Fx ⊃ Ga} 12, DN
QED

7.

	1. ~(∃x)Ix	ACP
	2. (∀x)~Ix	1, QE
	3. ~(∀x)(Ix ⊃ Jx)	AIP
	4. (∃x)~(Ix ⊃ Jx)	3, QE
	5. ~(Ia ⊃ Ja)	4, EI
	6. ~(~Ia ∨ Ja)	5, impl
	7. ~ ~Ia • ~Ja	6, DM
	8. ~ ~Ia	7, simp
	9. Ia	8, DN
	10. ~Ia	3, UI
	11. Ia • ~Ia	9, 10 conj
	12. ~ ~(∀x)(Ix ⊃ Jx)	3-11, IP
	13. (∀x)(Ix ⊃ Jx)	12, DN
	14. ~(∃x)Ix ⊃ (∀x)(Ix ⊃ Jx)	1-13 CP
	15. ~ ~(∃x)Ix ∨ (∀x)(Ix ⊃ Jx)	14, impl
	16. (∃x)Ix ∨ (∀x)(Ix ⊃ Jx)	15, DN
	QED	

8.

	1. (∀x)Dx ∨ (∀x)Ex	ACP
	2. ~(∀x)(Dx ∨ Ex)	AIP
	3. (∃x)~(Dx ∨ Ex)	2, QE
	4. ~(Da ∨ Ea)	3, EI
	5. ~Da • ~Ea	4, DM
	6. ~Da	5, simp
	7. (∃x)~Dx	6, EG
	8. ~(∀x)Dx	7, QE
	9. (∀x)Ex	1, 8, DS
	10. Ea	9, UI
	11. ~Ea • ~Da	5, com
	12. ~Ea	11, simp
	13. Ea • ~Ea	10, 12, conj
	14. ~ ~(∀x)(Dx ∨ Ex)	2-13 IP
	15. (∀x)(Dx ∨ Ex)	14, DN
	16. [(∀x)Dx ∨ (∀x)Ex] ⊃ (∀x)(Dx ∨ Ex)	1-15 CP
	QED	

```
9.          │1. (∃x)Ax ⊃ Ba                           ACP
            │      │2. ~(∀x)(Ax ⊃ Ba)                 AIP
            │      │3. (∃x)~(Ax ⊃ Ba)                 2, QE
            │      │4. ~(Ab ⊃ Ba)                     3, EI
            │      │5. ~(~Ab ∨ Ba)                    4, impl
            │      │6. ~~Ab • ~Ba                     5, DM
            │      │7. Ab • ~Ba                       6, DN
            │      │8. Ab                             7, simp
            │      │9. (∃x)Ax                         8, EG
            │      │10. Ba                            1, 9, MP
            │      │11. ~Ba • Ab                      7, com
            │      │12. ~Ba                           11, simp
            │      │13. Ba • ~Ba                      10, 12, conj
            │14. ~~(∀x)(Ax ⊃ Ba)                      2-13, IP
            │15. (∀x)(Ax ⊃ Bx)                        14, DN
     16. [(∃x)Ax ⊃ Ba] ⊃ (∀x)(Ax ⊃ Bx)           1-15, CP
            │17. (∀x)(Ax ⊃ Ba)                        ACP
            │      │18. (∃x)Ax                        ACP
            │      │19. Ab                            16, EI
            │      │20. Ab ⊃ Ba                       17, UI
            │      │21. Ba                            19, 20 MP
            │22. (∃x)Ax ⊃ Ba                          18-21 CP
     23. (∀x)(Ax ⊃ Ba) ⊃ [(∃x)Ax ⊃ Ba]           17-22 CP
     24. {[(∃x)Ax ⊃ Ba] ⊃ (∀x)(Ax ⊃ Bx)} • {(∀x)(Ax ⊃ Ba) ⊃ [(∃x)Ax ⊃ Ba]}     16, 23, conj
     25. [(∃x)Ax ⊃ Ba] ≡ (∀x)(Ax ⊃ Bx)           24, equiv
            QED
```

10.

│1. (∃x)(Ka • Lx)	ACP
│ │2. ~[Ka • (∃x)Lx]	AIP
│ │3. ~Ka ∨ ~(∃x)Lx	2, DM
│ │4. Ka • Lb	1, EI
│ │5. Ka	4, simp
│ │6. ~ ~Ka	5, DN
│ │7. ~(∃x)Lx	3, 6 DS
│ │8. (∀x)~Lx	7, QE
│ │9. ~Lb	8, UI
│ │10. Lb • Ka	4, com
│ │11. Lb	10, simp
│ │12. ~Lb • Lb	9, 11 conj
│13. ~ ~[Ka • (∃x)Lx]	2-12 IP
│14. Ka • (∃x)Lx	13, DN
15. (∃x)(Ka • Lx) ⊃ [Ka • (∃x)Lx]	1-14 CP
│16. Ka • (∃x)Lx	ACP
│ │17. ~(∃x)(Ka • Lx)	AIP
│ │18. (∀x)~(Ka • Lx)	17, QE
│ │19. ~(Ka • Lx)	18, UI
│ │20. ~Ka ∨ ~Lx	19, DM
│ │21. Ka	16, simp
│ │22. ~ ~Ka	21, DN
│ │23. ~Lx	20, 22 MP
│ │24. (∀x)~Lx	23, UG
│ │25. ~(∃x)Lx	24, QE
│ │26. (∃x)Lx • Ka	16, com
│ │27. (∃x)Lx	26, simp
│ │28. ~(∃x)Lx • (∃x)Lx	24, 27 conj
│29. ~ ~(∃x)(Ka • Lx)	17-28 IP
│30. (∃x)(Ka • Lx)	29, DN
31. [Ka • (∃x)Lx] ⊃ (∃x)(Ka • Lx)	16-30 CP
32. {(∃x)(Ka • Lx) ⊃ [Ka • (∃x)Lx]} • {[Ka • (∃x)Lx] ⊃ (∃x)(Ka • Lx)}	15, 31 conj
33. (∃x)(Ka • Lx) ≡ [Ka • (∃x)Lx]	32, equiv

QED

Exercises 3.11

1. Sa • Ial • (∀x)[(Sx • Ixl • x≠a) ⊃ Bax]
2. ~Icb • (∀x)[(Px • x≠b) ⊃ Icx]
3. (∃x)(∃y)(Sx • Sy • x≠y)
4. (∃x)(∃y){(Sx • Sy • x≠y) • (∀z)[Sz ⊃ (z=x ∨ z=y)]}
5. (∃x){(Mx • Hx) • (∀y)[(My • Hy) ⊃ y=x]}
6. (∃x)(∃y){Mx • Hx • My • Hy • x≠y • (∀z)[(Mz • Hz) ⊃ z=x]}
7. (∃x){Vx • (∀y)[(Vy ⊃ y=x) • x=d]}
8. (∀x)(∀y)[(Px • Axr • Gxh • Py • Ayr • Gxh) ⊃ x=y]
9. (∀x)(∀y)(∀z)[(Px • Axr • Gxh • Py • Ayr • Gyh • Pz • Azr • Gzh) ⊃ (x=y ∨ x=z ∨ y=z)]
10. (∀x)(∀y)(∀z)(∀w)[(Px • Axr • Gxh • Py • Ayr • Gyh • Pz • Azr • Gzh • Pw • Awr • Gwh) ⊃ (x=y ∨ x=z ∨ x=w ∨ y=z ∨ y=w ∨ z=w)]
11. Tcd • (∀x)(Txd ⊃ x=c)
12. Tcd • (∀x)(Tcx ⊃ x=d)
13. (∃x)(∃y)(Nx • Ixj • Ny • Iyj • x≠y)
14. (∃x)(∃y){Nx • Ixj • Ny • Iyj • x≠y • (∀z)[(Nz • Izj) ⊃ (z=x ∨ z=y)]}
15. (∃x)(∃y)(∃z){Nx • Ixj • Ny • Iyj • Nz • Izj • x≠y • x≠z • y≠z • (∀w)[(Nw • Iwj) ⊃ (w=x ∨ w=y ∨ w=z)]}
16. (∀x)(∀y)(∀z)[(Cx • Ixm • Cy • Iym • Cz • Izm) ⊃ (x=y ∨ x=z ∨ y=z)]
17. (∃x)(∃y){[Cx • Ixm • Cy • Iym • x≠y] • (∀z)[(Cz • Izm) ⊃ (z=x ∨ z=y)]
18. Sm • Ims • (∀x)[(Sx • Ixs • x≠m) ⊃ Bmx]
19. Sg • (∃x)(Ix • Sgx) • (∀x){[Sx • (∃y)(Iy • Sxy)] ⊃ x=g}
20. Pe • ~Seg • (∀x)[(Px • x≠e) ⊃ Sxg)]
21. (∃x)[Oxg • (∀y)(Oyg ⊃ y=x) • Rx]
22. (∃x)(∃y)(∃z)(Wx • Sxf • Wy • Syf • Wz • Szf • x≠y • x≠z • y≠z)
23. Wf • (∀x)[(Wx • x≠f) ⊃ Bfx]
24. Wg • Igc • (∀x)[(Wx • Ixc • x≠g) ⊃ Sgx]
25. (∃x)(Hx • Lgx) • (∀x)[(∃y)(Hy • Lxy) ⊃ x=g]
26. Ph • ~Lhg • (∀x)[(Px • x≠h) ⊃ Lxg]
27. (∃x)(∃y)(Wx • Exi • Wy • Eyi • x≠y)
28. (∃x)(∃y){Wx • Exi • Wy • Eyi • x≠y • (∀z)[(Wz • Ezi) ⊃ (z=x ∨ z=y)]}
29. (∃x){(Sx • Ax) • (∀y)[(Sy • Ay) ⊃ y=x] • x=n}
30. (∀x)(∀y)(∀z)[(Sx • Fxn • Sy • Fyn • Sz • Fzn) ⊃ (x=y ∨ x=z ∨ y=z)]
31. (∀x)(∀y){{Nx • Ny • (∃z){Az • Hxz • (∃w)[(Aw • Hmx) • Bzw]} • (∃z){Az • Hyz • (∃w)[(Aw • Hmx) • Bzw]}} ⊃ x=y}
32. (∀x)(∀y)(∀z){{Nx • Ny • Nz • (∃w){Aw • Hxw • (∃v)[(Av • Hmv) • Bwv]} • (∃w){Aw • Hy • (∃v)[(Av • Hmv) • Bwv]} • (∃w){Aw • Hzw • (∃v)[(Av • Hmv) • Bwv]}} ⊃ (x=y ∨ x=z ∨ y=z)}
33. Tec
34. Be • Ien • (∀x)[(Bx • Ixn • x≠e) ⊃ Tex]
35. Pk • ~Hkn • (∀x)[(Px • x≠k) ⊃ Hxn]
36. Pk • ~Hkn • Pa • ~Han • (∀x)[(Px • x≠k • x≠a) ⊃ Hxn]
37. (∃x)(∃y)(Px • Sxl • Py • Syl • x≠y)
38. (∃x)(∃y)(∃z)(Px • Sxl • Py • Syl • Pz • Szl • x≠y • x≠z • y≠z)
39. (∃x){(Tx • Sx • Ixp) • (∀y)[(Ty • Sy • Iyp) ⊃ y=x]}
40. (∃x){Lx • (∀y)[(Ly ⊃ y=x) • Fx]}
41. Fd • (∃x)(Ax • Wsxd) • (∀y){[Fy • (∃z)(Az • Wszy)] ⊃ y=d}
42. (∃x){(Tx • Sx • Wxa) • (∀y)[(Ty • Sy • Wya) ⊃ y=x]}
43. Pl • ~Wla • (∀x)[(Px • x≠l) ⊃ Wxa]
44. (∃x)(∃y)(∃z)[Sx • (∃w)(Bw • Exw) • Sy • (∃w)(Bw • Eyw) • Sz • (∃w)(Bw • Ezw) • x≠y • x≠z • y≠z]
45. (∀x)(∀y)(∀z){[Sx • (∃w)(Tw • Ixw) • Sy • (∃w)(Tw • Iyw) • Sz • (∃w)(Tw • Ixw)] ⊃ (x=y ∨ x=z ∨ y=z)}
46. (∃x)[Qx • (∀y)(Qy ⊃ y=x) • Bx]
47. We • Ieb • (∀x)[(Wx • Ixb • x≠e) ⊃ Pex]
48. (∃x)(∃y){Sxe • Sye • x≠y • (∀z)[Sze ⊃ (z=x ∨ z=y)]}
49. (∀x)(∀y)[(Qx • Ixe • Qy • Iye) ⊃ x=y]
50. (∀x){Sx ⊃ (∃y){(Cy • Oyx) • (∀z)[(Cz • Ozx) ⊃ z=y]}}
51. St • (∀x)[(Sx • x≠t) ⊃ Btx]

52. (∃x)(∃y)[Sx • Ixl • Sy • Iyl • x≠y • (∃z)(Pz • Wtz • Rxz • Ryz)]

53. $(\forall x)[(Sx \bullet Ixl \bullet x \neq m) \supset (\exists y)(Py \bullet Wty \bullet Rxy)]$
54. $(\forall x)\{(\exists y)[Gy \bullet Rxy \bullet (\exists z)(Gz \bullet Rjz \bullet Hyz)] \supset (x=n \vee x=r)\}$
55. $Bp \bullet \sim Tpc \bullet (\forall x)[(Bx \bullet x \neq p) \supset Txc]$
56. $(\exists x)(\exists y)(Sx \bullet Ixp \bullet Sy \bullet Iyp \bullet x \neq y)$
57. $(\exists x)(\exists y)(\exists z)(Sx \bullet Ixp \bullet Sy \bullet Iyp \bullet Sz \neq Izp \bullet x \neq y \bullet x \neq z \bullet y \neq z)$
58. $(\forall w)(\forall x)(\forall y)(\forall z)[(Pw \bullet Iwp \bullet Px \bullet Ixp \bullet Py \bullet Iyp \bullet Pz \bullet Lzp) \supset (w=x \vee w=y \vee w=z \vee x=y \vee x=z \vee y=z)]$
59. $Ps \bullet (\exists x)\{(Cx \bullet Hsx) \bullet (\forall y)\{(Py \bullet y \neq s) \supset (\forall z)[(Cz \bullet Hyz) \supset Bxz]\}\}$
60. $(\exists x)(\exists y)\{Px \bullet Txl \bullet Py \bullet Tyl \bullet x \neq y \bullet (\forall z)[(Pz \bullet Tzl) \supset (z=x \vee z=y)]\}$

Exercises 3.12a

1.

1. Dkm • (∀x)(Dkx ⊃ x=m)	
2. Dab	
3. Fb • ~Fm	/ a≠k
4. (∀x)(Dkx ⊃ x=m) • Dkm	1, Com
5. (∀x)(Dkx ⊃ x=m)	4, Simp
| 6. a=k	AIP
| 7. Dkb	2, 6, IDi
| 8. Dkb ⊃ b=m	5, UI
| 9. b=m	8, 7, MP
| 10. Fb	3, Simp
| 11. Fm	10, 9, IDi
| 12. ~Fm • Fb	3, Com
| 13. ~Fm	12, Simp
| 14. Fm • ~Fm	11, 13, Conj
15. a≠k	6-14, IP

QED

2.

1. (∃x)(∃y)[(Fx • Fy) ⊃ x=y]	
2. (∀x)(∀y)x≠y	
3. (∃y)[(Fa • Fy) ⊃ a=y]	1, EI
4. (Fa • Fb) ⊃ a=b	3, EI
5. (∀y)a≠y	2, UI
6. a≠b	5, UI
7. ~(Fa • Fb)	4, 6 MT
8. ~Fa ∨ ~Fb	7, DM
9. Fa ⊃ ~Fb	8, impl

QED

3.

1. (∀x)[(∃y)Pxy ⊃ (∃z)Pzx]	
2. (∃x)(Pxb • x=d)	
3. Pab • a=d	2, EI
4. (∃y)Pay ⊃ (∃z)Pza	1, UI
5. Pab	3, simp
6. (∃y)Pay	5, EG
7. (∃z)Pza	4, 6 MP
8. a=d • Pab	3, Com
9. a=d	8, simp
10. (∃z)Pzd	7, 10, IDi

QED

4.

1. (∀x)[Jx ∨ (Kx • Lx)]	
2. ~(Ja ∨ Kb)	
| 3. a=b	AIP
| 4. ~Ja • ~Kb	2, DM
| 5. ~Ja	4, simp
| 6. Ja ∨ (Ka • La)	1, UI
| 7. Ka • La	5,6 DS
| 8. Ka	7, simp
| 9. ~Kb • ~Ja	4, Com
| 10. ~Kb	9, Simp
| 11. ~Ka	10, 3, IDi
| 12. Ka • ~Ka	8, 11, conj
13. a≠b	3-12 IP

QED

5.

1. (∀x)[(Mx ∨ Nx) ⊃ Ox]	

2. $\sim$Oc
3. Md

	4. c=d	AIP
	5. (Mc ∨ Nc) ⊃ Oc	1, UI
	6. $\sim$(Mc ∨ Nc)	2, 5 MT
	7. $\sim$Mc • $\sim$Nc	6, DM
	8. $\sim$Mc	7, simp
	9. $\sim$Md	4, 8 IDi
	10. Md • $\sim$Md	3, 9 conj
11. c≠d		4-10 IP

QED

6.

1. (∀x)(Qx ⊃ Sx)	
2. (∀x)(Rx ⊃ Tx)	
3. (∀x)[Qx ∨ (Rx • Ux)]	
4. a=b	
5. Qa ⊃ Sa	1, UI
6. Ra ⊃ Ta	2, UI
7. (Qa ⊃ Sa) • (Ra ⊃ Ta)	5, 6 conj
8. Qa ∨ (Ra • Ua)	3, UI
9. (Qa ∨ Ra) • (Qa ∨ Ua)	8, dist
10. Qa ∨ Ra	9, simp
11. Sa ∨ Ta	7, 10 CD
12. Sb ∨ Ta	4, 11 IDi

QED

7.

1. Fac • Fbc • (∀x)[Fxc ⊃ (x=a ∨ x=b)]	
2. (∃x)(Fxc • x≠a)	
3. Fb • Gb	
4. Fdc • d≠a	2, EI
5. (∀x)[Fxc ⊃ (x=a ∨ x=b)]	1, simp
6. Fdc ⊃ (d=a ∨ d=b)	5, UI
7. Fdc	4, simp
8. d=a ∨ d=b	6, 7 MP
9. d≠a	4, simp
10. d=b	8, 9 DS
11. b=d	10, IDs
12. Fd • Gd	3, 11 IDi

QED

8.

1. (∀x)(∀y)[Ax ⊃ (By ⊃ Cxy)]	
2. Aa • Ba	
3. a=b	/ Cab
4. (∀y)[Aa ⊃ (By ⊃ Cay)]	1, UI
5. Aa ⊃ (Ba ⊃ Caa)	4, UI
6. Aa	2, simp
7. Ba ⊃ Caa	5, 6, MP
8. Ba • Aa	2, Com
9. Ba	9, simp
10. Caa	7, 9, MP
11. Cab	10, 3, IDi

QED

9.
1. (∃x)(Mx • Px)
2. (∀x)[Mx ⊃ (∀y)(Ky ⊃ x=y)]
3. Kf / Mf • Pf
4. Ma • Pa 1, EI
5. Ma 4, Simp
6. Ma ⊃ (∀y)(Ky ⊃ a=y) 2, UI
7. (∀y)(Ky ⊃ a=y) 6, 5, MP
8. Kf ⊃ a=f 7, UI
9. a=f 8, 3, MP
10. Mf • Pf 4, 9, IDi

QED

10.
1. (∀x)[Ax ∨ (Bx • Cx)]
2. ~(∀x)Bx
3. (∀x)(Ax ⊃ x=c) / (∃x)x=c
4. (∃x)~Bx 2, QE
5. ~Ba 4, EI
6. ~Ba ∨ ~Ca 5, Add
7. ~(Ba • Ca) 6, DM
8. Aa ∨ (Ba • Ca) 1, UI
9. (Ba • Ca) ∨ Aa 8, Com
10. Aa 9, 7, DS
11. Aa ⊃ a=c 3, UI
12. a=c 11, 10, MP
13. (∃x)x=c 12, EG

QED

11.
1. Dp • (∃x)(Ex • ~Fxp)
2. (∀x)[Gx ⊃ (∀y)Fyx] / (∃x)(Dx • ~Gx)
| 3. (∀x)(Dx ⊃ Gx) AIP
| 4. Dp ⊃ Gp 3, UI
| 5. Dp 1, Simp
| 6. Gp 4, 5, MP
| 7. Gp ⊃ (∀y)Fyp 2, UI
| 8. (∀y)Fyp 7, 6, MP
| 9. (∃x)(Ex • ~Fxp) 1, Simp
| 10. Ea • ~Fap 9, EI
| 11. ~Fap • Ea 10, Com
| 12. ~Fap 11, Simp
| 13. Fap 8, UI
| 14. Fap • ~Fap 13, 12, Conj
15. ~(∀x)(Dx ⊃ Gx) 3-14, IP
16. (∃x)~(Dx ⊃ Gx) 15, QE
17. (∃x)~(~Dx ∨ Gx) 16, Impl
18. (∃x)(~~Dx • ~Gx) 17, DM
19. (∃x)(Dx • ~Gx) 18, DN

QED

12.
1. Ha • Ia • (∀x)[(Hx • Ix) ⊃ x=a]
2. Hb • Jb • (∀x)[(Hx • Jx) ⊃ x=b]
3. Ka • ~Kb   /~(∃x)(Hx • Ix • Jx)
| 4. (∃x)(Hx • Ix • Jx)   AIP
| 5. Hc • Ic • Jc   4, EI
| 6. (∀x)[(Hx • Ix) ⊃ x=a]   1, simp
| 7. (Hc • Ic) ⊃ c=a   6, UI
| 8. Hc • Ic   5, simp
| 9. c=a   7, 8, MP
| 10. (∀x)[(Hx • Jx) ⊃ x=b]   2, simp
| 11. (Hc • Jc) ⊃ c=b   10, UI
| 12. Hc • Jc   5, simp
| 13. c=b   11,12, MP
| 14. a=b   13, 9, IDi
| 15. Ka   3, Simp
| 16. Kb   15, 14, IDi
| 17. ~Kb   3, Simp
| 18. Kb • ~Kb   16, 17, Conj
19. ~(∃x)(Hx • Ix • Jx)   4-18, IP

QED

13.
1. La • Lb • a≠b
2. (∀x)(∀y)(∀z)[(Lx • Ly • Lz) ⊃ (x=y ∨ y=z ∨ x=z)]
/ (∀x)[Lx ⊃ (x=a ∨ x=b)]
| 3. Lx   ACP
| 4. (∀y)(∀z)[(La • Ly • Lz) ⊃ (a=y ∨ y=z ∨ a=z)]   2, UI
| 5. (∀z)[(La • Lb • Lz) ⊃ (a=b ∨ b=z ∨ a=z)]   4, UI
| 6. (La • Lb • Lx) ⊃ (a=b ∨ b=x ∨ a=x)   5, UI
| 7. La • Lb   1, Simp
| 8. La • Lb • Lx   7, 3, Conj
| 9. a=b ∨ b=x ∨ a=x   6, 8, MP
| 10. a≠b   1, Simp
| 11. b=x ∨ a=x   9, 10, DS
| 12. a=x ∨ b=x   11, Com
| 13. x=a ∨ x=b   12, IDs
14. Lx ⊃ (x=a ∨ x=b)   3-13, CP
15. (∀x)[Lx ⊃ (x=a ∨ x=b)]   14, UG

QED

14.
1. (∀x)(Ecx ⊃ x=d)
2. (∀x){(Fx • Gx) ⊃ (∀y)[(Fy • Gy) ⊃ y=x]}
3. (∃x)(Fx • Gx • Ecx)
4. Fa • Ga   / a=d
5. (Fa • Ga) ⊃ (∀y)[(Fy • Gy) ⊃ y=a]   2, UI
6. (∀y)[(Fy • Gy) ⊃ y=a]   5, 4, MP
7. Fb • Gb • Ecb   3, EI
8. Fb • Gb   7, Simp
9. (Fb • Gb) ⊃ b=a   6, UI
10. b=a   9, 8, MP
11. Ecb ⊃ b=d   1, UI
12. Ecb   7, Simp
13. b=d   11, 12, MP
14. a=d   13, 10, IDi

QED

15.	1. $(\exists x)(\exists y)(Hx \bullet Ix \bullet Jx \bullet Hy \bullet Iy \bullet Jy \bullet x{\neq}y)$	
	2. $(\forall x)(\forall y)[(Hx \bullet Ix \bullet Jx \bullet Hy \bullet Iy \bullet Jy) \supset x{=}y]$	
	3. $(\exists y)(Ha \bullet Ia \bullet Ja \bullet Hy \bullet Iy \bullet Jy \bullet a{\neq}y)$	1, EI
	4. $Ha \bullet Ia \bullet Ja \bullet Hb \bullet Ib \bullet Jb \bullet a{\neq}b$	3, EI
	5. $\sim(\forall z)[(Hz \bullet Iz \bullet Jz) \supset (z{=}a \vee z{=}b)]$	AIP
	6. $(\exists z)\sim[(Hz \bullet Iz \bullet Jz) \supset (z{=}a \vee z{=}b)]$	5, QE
	7. $\sim[(Hc \bullet Ic \bullet Jc) \supset (c{=}a \vee c{=}b)]$	6, EI
	8. $\sim[\sim(Hc \bullet Ic \bullet Jc) \vee (c{=}a \vee c{=}b)]$	7, impl
	9. $\sim\sim(Hc \bullet Ic \bullet Jc) \bullet \sim(c{=}a \vee c{=}b)$	8, DM
	10. $(Hc \bullet Ic \bullet Jc) \bullet \sim(c{=}a \vee c{=}b)$	9, DN
	11. $Hc \bullet Ic \bullet Jc \bullet c{\neq}a \bullet c{\neq}b$	10, DM
	12. $(\forall y)[(Ha \bullet Ia \bullet Ja \bullet Hy \bullet Iy \bullet Jy) \supset a{=}y]$	2, UI
	13. $(Ha \bullet Ia \bullet Ja \bullet Hc \bullet Ic \bullet Jc) \supset a{=}c$	12, UI
	14. $Ha \bullet Ia \bullet Ja$	4, simp
	15. $Hc \bullet Ic \bullet Jc$	11, simp
	16. $Ha \bullet Ia \bullet Ja \bullet Hc \bullet Ic \bullet Jc$	14, 15 conj
	17. $a{=}c$	13, 16 MP
	18. $c{=}a$	17, IDs
	19. $c{\neq}a$	11, simp
	20. $c{=}a \bullet c{\neq}a$	18, 19 conj
	21. $\sim\sim(\forall z)[(Hz \bullet Iz \bullet Jz) \supset (z{=}a \vee z{=}b)]$	5-20 IP
	22. $(\forall z)[(Hz \bullet Iz \bullet Jz) \supset (z{=}a \vee z{=}b)]$	21, DN
	23. $Ha \bullet Ia \bullet Ja \bullet Hb \bullet Ib \bullet Jb \bullet a{\neq}b \bullet (\forall z)[(Hz \bullet Iz \bullet Jz) \supset (z{=}a \vee z{=}b)]$	4, 22 conj
	24. $(\exists y)\{Ha \bullet Ia \bullet Ja \bullet Hy \bullet Iy \bullet Jy \bullet a{\neq}y \bullet (\forall z)[(Hz \bullet Iz \bullet Jz) \supset (z{=}a \vee z{=}y)]\}$	23, EG
	25. $(\exists x)(\exists y)\{Hx \bullet Ix \bullet Jx \bullet Hy \bullet Iy \bullet Jy \bullet x{\neq}y \bullet (\forall z)[(Hz \bullet Iz \bullet Jz) \supset (z{=}x \vee z{=}y)]\}$	24, EG
	QED	

16.

1. Na • Oa • Nb • Ob • a≠b • (∀x)[(Nx • Ox) ⊃ (x=a ∨ x=b)]	
2. Na • ~Pa • (∀x)[(Nx • x≠a) ⊃ Px]	/ (∃x){Nx • Ox • Px • (∀y)[(Ny • Oy • Py) ⊃ y=x]}
│3. ~(∀y)[(Ny • Oy • Py) ⊃ y=b]	AIP
│4. (∃y)~[(Ny • Oy • Py) ⊃ y=b]	3, QE
│5. ~[(Nc • Oc • Pc) ⊃ c=b]	4, EI
│6. ~[~(Nc • Oc • Pc) ∨ c=b]	5, impl
│7. ~~(Nc • Oc • Pc) • c≠b	6, DM
│8. Nc • Oc • Pc • c≠b	7, DN
│9. (∀x)[(Nx • Ox) ⊃ (x=a ∨ x=b)]	1, simp
│10. (Nc • Oc) ⊃ (c=a ∨ c=b)	9, UI
│11. Nc • Oc	8, simp
│12. c=a ∨ c=b	10, 11, MP
│13. c≠b	8, simp
│14. c=a	12, 13, DS
│15. ~Pa	2, simp
│16. a=c	14, IDs
│17. ~Pc	15, 16, IDi
│18. Pc	8, simp
│19. Pc • ~Pc	18, 17, conj
20. ~~(∀y)[(Ny • Oy • Py) ⊃ y=b]	3-19, IP
21. (∀y)[(Ny • Oy • Py) ⊃ y=b]	20, DN
22. (∀x)[(Nx • x≠a) ⊃ Px]	2, simp
23. (Nb • b≠a) ⊃ Pb	22, UI
24. Nb	1, simp
25. a≠b	1, Simp
26. b≠a	25, IDs
27. Nb • b≠a	24, 26, conj
28. Pb	23, 27, MP
29. Nb • Ob	1, simp
30. Nb • Ob • Pb	29, 28, conj
31. Nb • Ob • Pb • (∀y)[(Ny • Oy • Py) ⊃ y=b]	30, 22, Conj
32. (∃x){Nx • Ox • Px • (∀y)[(Ny • Oy • Py) ⊃ y=x]}	31, EG

QED

17.

1.	$(\exists x)(\exists y)(Kx \bullet Lx \bullet Ky \bullet Ly \bullet x \neq y)$	
2.	$Ka \bullet La \bullet Ma \bullet (\forall y)[(Ky \bullet Ly \bullet My) \supset y{=}a]$	
3.	$\sim(\exists x)(Kx \bullet Lx \bullet \sim Mx)$	AIP
4.	$(\forall x)\sim(Kx \bullet Lx \bullet \sim Mx)$	3, QE
5.	$(\exists y)(Kb \bullet Lb \bullet Ky \bullet Ly \bullet b \neq y)$	1, EI
6.	$Kb \bullet Lb \bullet Kc \bullet Lc \bullet b \neq c$	5, EI
7.	$(\forall y)[(Ky \bullet Ly \bullet My) \supset y{=}a]$	2, simp
8.	$(Kb \bullet Lb \bullet Mb) \supset b{=}a$	7, UI
9.	$\sim(Kb \bullet Lb \bullet \sim Mb)$	4, UI
10.	$\sim Kb \vee \sim Lb \vee \sim\sim Mb$	9, DM
11.	$\sim Kb \vee \sim Lb \vee Mb$	10, DN
12.	Kb	6, simp
13.	$\sim\sim Kb$	12, DN
14.	$\sim Lb \vee Mb$	11, 13 DS
15.	Lb	6, simp
16.	$\sim\sim Lb$	15, DN
17.	Mb	14, 16 DS
18.	$Kb \bullet Lb \bullet Mb$	12, 15, 17 conj
19.	$b{=}a$	8, 18 MP
20.	$(Kc \bullet Lc \bullet Mc) \supset c{=}a$	7, UI
21.	$\sim(Kc \bullet Lc \bullet \sim Mc)$	4, UI
22.	$\sim Kc \vee \sim Lc \vee \sim\sim Mc$	21, DM
23.	$\sim Kc \vee \sim Lc \vee Mc$	22, DN
24.	Kc	6, simp
25.	$\sim\sim Kc$	24, DN
26.	$\sim Lc \vee Mc$	23, 25 DS
27.	Lc	6, simp
28.	$\sim\sim Lc$	27, DN
29.	Mc	26, 28 DS
30.	$Kc \bullet Lc \bullet Mc$	24, 27, 29 conj
31.	$c{=}a$	20, 30 MP
32.	$a{=}c$	31, IDs
33.	$b{=}c$	19, 32 IDi
34.	$b \neq c$	6, simp
35.	$b{=}c \bullet b \neq c$	33, 34 conj
36.	$\sim\sim(\exists x)(Kx \bullet Lx \bullet \sim Mx)$	3-35 IP
37.	$(\exists x)(Kx \bullet Lx \bullet \sim Mx)$	26, DN

QED

18.	1. $(\exists x)(\exists y)(Ax \bullet Cx \bullet Ay \bullet Cy \bullet x \neq y)$	
	2. $(\forall x)(\forall y)(\forall z)[(Cx \bullet Cy \bullet Cz) \supset (x=y \vee x=z \vee y=z)]$	
	3. $(\exists x)(Bx \bullet {\sim}Ax)$	/ ${\sim}(\forall x)(Bx \supset Cx)$
	4. $(\exists y)(Aa \bullet Ca \bullet Ay \bullet Cy \bullet a \neq y)$	1, EI
	5. $Aa \bullet Ca \bullet Ab \bullet Cb \bullet a \neq b$	4, EI
	6. $Bc \bullet {\sim}Ac$	3, EI
	│ 7. $(\forall x)(Bx \supset Cx)$	AIP
	│ 8. $Bc \supset Cc$	7, UI
	│ 9. Bc	6, Simp
	│ 10. Cc	8, 9, MP
	│ 11. Ca	5, Simp
	│ 12. Cb	5, Simp
	│ 13. $Ca \bullet Cb$	11, 12, Conj
	│ 14. $Ca \bullet Cb \bullet Cc$	13, 10, Conj
	│ 15. $(\forall y)(\forall z)[(Ca \bullet Cy \bullet Cz) \supset (a=y \vee a=z \vee y=z)]$	2, UI
	│ 16. $(\forall z)[(Ca \bullet Cb \bullet Cz) \supset (a=b \vee a=z \vee b=z)]$	15, UI
	│ 17. $(Ca \bullet Cb \bullet Cc) \supset (a=b \vee a=c \vee b=c)$	16, UI
	│ 18. $a=b \vee a=c \vee b=c$	17, 14, MP
	│ 19. $a \neq b$	5, simp
	│ 20. $a=c \vee b=c$	18, 19, DS
	│ │ 21. $a=c$	AIP
	│ │ 22. Aa	5, simp
	│ │ 23. Ac	22, 21, IDi
	│ │ 24. ${\sim}Ac$	6, simp
	│ │ 25. $Ac \bullet {\sim}Ac$	23, 24 conj
	│ 26. $a \neq c$	21-25, IP
	│ 27. $b=c$	20, 26, DS
	│ 28. Ab	5, simp
	│ 29. Ac	28, 27, IDi
	│ 30. ${\sim}Ac$	6, simp
	│ 31. $Ac \bullet {\sim}Ac$	29, 30, conj
	32. ${\sim}(\forall x)(Bx \supset Cx)$	7-31, IP

QED

19.

	1. $(\exists x)(\exists y)(Qx \bullet Rx \bullet Qy \bullet Ry \bullet x \neq y)$	
	2. $(\forall x)(\forall y)(\forall z)[(Rx \bullet Sx \bullet Ry \bullet Sy \bullet Rz \bullet Sz) \supset (x=y \vee x=z \vee y=z)]$	
	3. $(\forall x)(\sim Qx \vee Sx)$	
	4. $(\exists y)(Qa \bullet Ra \bullet Qy \bullet Ry \bullet a \neq y)$	1, EI
	5. $Qa \bullet Ra \bullet Qb \bullet Rb \bullet a \neq b$	4, EI
	6. $\sim Qa \vee Sa$	3, UI
	7. Qa	5, simp
	8. $\sim\sim Qa$	7, DN
	9. Sa	6, 8 DS
	10. $\sim Qb \vee Sb$	3, UI
	11. Qb	5, simp
	12. $\sim\sim Qb$	11, DN
	13. Sb	10, 12 DS
	14. $Qa \bullet Ra \bullet Sa \bullet Qb \bullet Rb \bullet Sb \bullet a \neq b$	5, 9, 13 conj
	| 15. $\sim(\forall z)[(Rz \bullet Sz) \supset (z=a \vee z=b)]$	AIP
	| 16. $(\exists z)\sim[(Rz \bullet Sz) \supset (z=a \vee z=b)]$	15, QE
	| 17. $\sim[(Rc \bullet Sc) \supset (c=a \vee c=b)]$	16, EI
	| 18. $\sim[\sim(Rc \bullet Sc) \vee (c=a \vee c=b)]$	17, impl
	| 19. $\sim\sim(Rc \bullet Sc) \bullet \sim(c=a \vee c=b)$	18, DM
	| 20. $Rc \bullet Sc \bullet \sim(c=a \vee c=b)$	19, DN
	| 21. $Rc \bullet Sc \bullet c \neq a \bullet c \neq b$	20, DM
	| 22. $(\forall y)(\forall z)[(Ra \bullet Sa \bullet Ry \bullet Sy \bullet Rz \bullet Sz) \supset (a=y \vee a=z \vee y=z)]$	2, UI
	| 23. $(\forall z)[(Ra \bullet Sa \bullet Rb \bullet Sb \bullet Rz \bullet Sz) \supset (a=b \vee a=z \vee b=z)]$	22, UI
	| 24. $(Ra \bullet Sa \bullet Rb \bullet Sb \bullet Rc \bullet Sc) \supset (a=b \vee a=c \vee b=c)$	23, UI
	| 25. $Rc \bullet Sc$	21, simp
	| 26. $Ra \bullet Sa \bullet Rb \bullet Sb$	14, simp
	| 27. $Ra \bullet Sa \bullet Rb \bullet Sb \bullet Rc \bullet Sc$	25, 26 conj
	| 28. $a=b \vee a=c \vee b=c$	24, 27 MP
	| 29. $c \neq a$	21, simp
	| 30. $a \neq c$	29, IDs
	| 31. $a=b \vee b=c$	28, 30 DS
	| 32. $c \neq b$	21, simp
	| 33. $b \neq c$	32, IDs
	| 34. $a=b$	31, 34 DS
	| 35. $a \neq b$	5, simp
	| 36. $a=b \bullet a \neq b$	34, 35 conj
	37. $\sim\sim(\forall z)[(Rz \bullet Sz) \supset (z=a \vee z=b)]$	15-36 IP
	38. $(\forall z)[(Rz \bullet Sz) \supset (z=a \vee z=b)]$	37, DN
	39. $Qa \bullet Ra \bullet Sa \bullet Qb \bullet Rb \bullet Sb \bullet a \neq b \bullet (\forall z)[(Rz \bullet Sz) \supset (z=a \vee z=b)]$	14, 38 conj
	40. $(\exists y)\{Qa \bullet Ra \bullet Sa \bullet Qy \bullet Ry \bullet Sy \bullet a \neq y \bullet (\forall z)[(Rz \bullet Sz) \supset (z=a \vee z=y)]\}$	39, EG
	41. $(\exists x)(\exists y)\{Qx \bullet Rx \bullet Sx \bullet Qy \bullet Ry \bullet Sy \bullet x \neq y \bullet (\forall z)[(Rz \bullet Sz) \supset (z=x \vee z=y)]\}$	40, EG

QED

20.

1.	$Ma \bullet \sim Pa \bullet Mb \bullet \sim Pb \bullet (\forall x)[(Mx \bullet x \neq a \bullet x \neq b) \supset Px]$	
2.	$Qb \bullet (\forall x)[(Mx \bullet Qx) \supset x=b]$	
3.	$(\forall x)\{Mx \supset [\sim(Qx \lor Px) \equiv Rx]\}$	
4.	$a \neq b$	/ $(\exists x)\{Mx \bullet Rx \bullet (\forall y)[(My \bullet Ry) \supset y=x]\}$
5.	$Ma \supset [\sim(Qa \lor Pa) \equiv Ra]$	3, UI
6.	Ma	1, simp
7.	$\sim(Qa \lor Pa) \equiv Ra$	5, 6 MP
8.	$[\sim(Qa \lor Pa) \supset Ra] \bullet [Ra \supset \sim(Qa \lor Pa)]$	7, equiv
9.	$\sim(Qa \lor Pa) \supset Ra$	8, simp
10.	$(\forall x)[(Mx \bullet Qx) \supset x=b]$	2, simp
11.	$(Ma \bullet Qa) \supset a=b$	10, UI
12.	$\sim(Ma \bullet Qa)$	4, 11 MT
13.	$\sim Ma \lor \sim Qa$	12, DM
14.	$\sim\sim Ma$	6, DN
15.	$\sim Qa$	13, 14 DS
16.	$\sim Pa$	1, simp
17.	$\sim Qa \bullet \sim Pa$	15, 16 conj
18.	$\sim(Qa \lor Pa)$	17, DM
19.	Ra	9, 18 MP
20.	$Ma \bullet Ra$	6, 19 conj
21.	$\sim(\forall y)[(My \bullet Ry) \supset y=a]$	AIP
22.	$(\exists y)\sim[(My \bullet Ry) \supset y=a]$	21, QE
23.	$\sim[(Mc \bullet Rc) \supset c=a]$	22, EI
24.	$\sim[\sim(Mc \bullet Rc) \lor c=a]$	23, impl
25.	$\sim\sim(Mc \bullet Rc) \bullet c \neq a$	24, DM
26.	$Mc \bullet Rc \bullet c \neq a$	25, DN
27.	$Mc \supset [\sim(Qc \lor Pc) \equiv Rc]$	3, UI
28.	Mc	26, simp
29.	$\sim(Qc \lor Pc) \equiv Rc$	27, 28 MP
30.	$[\sim(Qc \lor Pc) \supset Rc] \bullet [Rc \supset \sim(Qc \lor Pc)]$	29, equiv
31.	$Rc \supset \sim(Qc \lor Pc)$	30, simp
32.	Rc	26, simp
33.	$\sim(Qc \lor Pc)$	31, 32 MP
34.	$\sim Qc \bullet \sim Pc$	33, DM
35.	$(\forall x)[(Mx \bullet x \neq a \bullet x \neq b) \supset Px]$	1, simp
36.	$(Mc \bullet c \neq a \bullet c \neq b) \supset Pc$	35, UI
37.	$\sim Pc$	34, simp
38.	$\sim(Mc \bullet c \neq a \bullet c \neq b)$	36, 37 MT
39.	$\sim Mc \lor \sim c \neq a \lor \sim c \neq b$	38, DM
40.	$\sim Mc \lor c=a \lor \sim c \neq b$	39, DM
41.	$\sim Mc \lor c=a \lor c=b$	40, DN
42.	$\sim\sim Mc$	28, DN
43.	$c=a \lor c=b$	41, 42 DS
44.	$c \neq a$	26, simp
45.	$c=b$	43, 44 DS
46.	Qb	2, simp
47.	$b=c$	45, IDs
48.	Qc	46, 47 IDi
49.	$\sim Qc$	34, simp
50.	$Qc \bullet \sim Qc$	48, 49 conj
51.	$\sim\sim(\forall y)[(My \bullet Ry) \supset y=a]$	21-50 IP
52.	$(\forall y)[(My \bullet Ry) \supset y=a]$	51, DN
53.	$Ma \bullet Ra \bullet (\forall y)[(My \bullet Ry) \supset y=a]$	20, 52 conj
54.	$(\exists x)\{Mx \bullet Rx \bullet (\forall y)[(My \bullet Ry) \supset y=x]\}$	53, EG

QED

Exercises 3.12b

1.

1. Fp	
2. $\sim Fo$	/ $p \neq o$
	3. $p=o$	AIP
	4. Fo	1, 3, IDi
	5. $\sim Fo \bullet Fo$	4, 2, conj
6. $p \neq o$	3-5, IP

QED

2.

1. $g=m \supset Mmw$	
2. $m=g$	
3. $w=h$	/ Mgh
4. $g=m$	1, IDs
5. Mmw	1, 4, MP
6. Mgw	5, 2, Idi
7. Mgh	6, 3, IDi

QED

3.

1. $(\exists x){\sim}Sx \supset (\forall x)x \neq w$	/ Sw
	2. $\sim Sw$	AIP
	3. $(\exists x){\sim}Sx$	2, EG
	4. $(\forall x)x \neq w$	1, 3, MP
	5. $w \neq w$	4, UI
	6. $w=w$	IDr
	7. $w=w \bullet w \neq w$	6, 5, conj
8. $\sim\sim Sw$	2-7, IP
9. Sw	8, DN

QED

4.

1. $Rk \bullet Tk \bullet (\forall x)[(Rx \bullet Tx \bullet x \neq k) \supset Fkx]$	
2. $Rp \bullet Tp$	
3. $k \neq p$	/ Fkp
4. $(\forall x)[(Rx \bullet Tx \bullet x \neq k) \supset Fkx]$	1, simp
5. $(Rp \bullet Tp \bullet p \neq k) \supset Fkp$	4, UI
6. $p \neq k$	3, IDs
7. $Rp \bullet Tp \bullet p \neq k$	2, 6, conj
8. Fkp	5, 7, MP

QED

5.

1. $Pr \bullet Cr \bullet Trm \bullet (\forall x)[(Tx \bullet Cx \bullet Txm) \supset x=r]$	
2. $(\forall x)[(Ex \bullet Px) \supset Wx]$	
3. $(\forall x)[(Px \bullet Cx) \supset Ex]$	Wr
4. $Pr \bullet Cr$	1, Simp
5. $(Pr \bullet Cr) \supset Er$	3, UI
6. Er	5, 4, MP
7. Pr	4, Simp
8. $Er \bullet Pr$	6, 7, Conj
9. $(Er \bullet Pr) \supset Wr$	2, UI
10. Wr	9, 8, MP

QED

6.	1. (∃x)[Wxr • (∀y)(Wyr ⊃ y=x) • Gx • Px]	
	2. Pl • ~Gl	/ ~Wlr
	| 3. Wlr	AIP
	| 4. War • (∀y)(Wyr ⊃ y=a) • Ga • Pa	1, EI
	| 5. (∀y)(Wyr ⊃ y=a)	4, simp
	| 6. Wlr ⊃ l=a	5, UI
	| 7. l=a	3, 6, MP
	| 8. Ga	4, simp
	| 9. Gl	7, 8, IDi
	| 10. ~Gl	2, simp
	| 11. Gl • ~Gl	9, 10, conj
	12. ~Wlr	3-11, IP
	QED	

7.	1. Pj • Sj • (∀y)[(Py • Sy) ⊃ y=j]	
	2. Pb • Cb • (∀y)[(Py • Cy) ⊃ y=b]	
	3. (∃x)(Px • Sx • Cx)	/ j=b
	4. Pa • Sa • Ca	3, EI
	5. (∀y)[(Py • Sy) ⊃ y=j]	1, simp
	6. (Pa • Sa) ⊃ a=j	5, UI
	7. Pa • Sa	4, simp
	8. a=j	6, 7, MP
	9. (∀y)[(Py • Cy) ⊃ y=b]	2, simp
	10. (Pa • Ca) ⊃ a=b	9, UI
	11. Pa • Ca	4, simp
	12. a=b	10, 11, MP
	13. j=b	12, 8, IDi
QED		

8.	1. Sp • ~Wp • (∀x)[(Sx • x≠p) ⊃ Wx]	
	2. Sr • ~Gr • (∀x)[Sx • x≠r) ⊃ Gx]	
	3. p≠r	/ Gp • Wr
	4. (∀x)[(Sx • x≠p) ⊃ Wx]	1, simp
	5. (Sr • r≠p) ⊃ Wr	4, UI
	6. r≠p	3, IDs
	7. Sr	2, simp
	8. Sr • r≠p	6, 7, conj
	9. Wr	5, 8, MP
	10. (∀x)[Sx • x≠r) ⊃ Gx]	2, simp
	11. (Sp • p≠r) ⊃ Gp	10, UI
	12. Sp	1, simp
	13. Sp • p≠r	3, 12, conj
	14. Gp	11, 13 MP
	15. Gp • Wr	9, 14, conj
	QED	

9.

1.	$(\exists x)\{Sx \bullet Gxs \bullet (\forall y)[(Sy \bullet Gys) \supset y=x]\}$	
2.	$(\exists x)(\exists y)(Sx \bullet Sy\ \ Gxl \bullet Gyl \bullet x \neq y)$	
3.	$\sim(\exists x)(Sx \bullet Gxs \bullet Gxl)$	$/ (\exists x)(\exists y)(\exists z)(Sx \bullet Sy \bullet Sz \bullet x \neq y \bullet x \neq z \bullet y \neq z)$
4.	$Sa \bullet Gas \bullet (\forall y)[(Sy \bullet Gys) \supset y=a)$	1, EI
5.	$(\exists y)(Sb \bullet Sy\ \ Gbl \bullet Gyl \bullet b \neq y)$	2, EI
6.	$Sb \bullet Sc \bullet Gbl \bullet Gcl \bullet b \neq c$	5, EI
7.	$(\forall x)\sim(Sx \bullet Gxs \bullet Gxl)$	3, QE
8.	$(\forall x)[\sim(Sx \bullet Gxs) \vee \sim Gxl]$	7, DM
9.	$(\forall x)[(Sx \bullet Gxs) \supset \sim Gxl]$	8, Impl
10.	$(Sa \bullet Gas) \supset \sim Gal$	9, UI
11.	$Sa \bullet Gas$	4, Simp
12.	$\sim Gal$	10, 11, MP
| 13.	$a=b$	AIP
| 14.	$\sim Gbl$	12, 13, IDi
| 15.	Gbl	6, Simp
| 16.	$Gbl \bullet \sim Gbl$	15, 14, Conj
17.	$a \neq b$	13-16, IP
| 18.	$a=c$	AIP
| 19.	$\sim Gcl$	12, 18, IDi
| 20.	Gcl	6, Simp
| 21.	$Gcl \bullet \sim Gcl$	20, 19, Conj
22.	$a \neq c$	18-21, IP
23.	Sa	4, Simp
24.	$Sb \bullet Sc$	6, Simp
25.	$Sa \bullet Sb \bullet Sc$	23, 24, Conj
26.	$Sa \bullet Sb \bullet Sc \bullet a \neq b$	25, 17, Conj
27.	$Sa \bullet Sb \bullet Sc \bullet a \neq b \bullet a \neq c$	26, 22, Conj
28.	$b \neq c$	6, Simp
29.	$Sa \bullet Sb \bullet Sc \bullet a \neq b \bullet a \neq c \bullet b \neq c$	27, 28, Conj
30.	$(\exists z)(Sa \bullet Sb \bullet Sz \bullet a \neq b \bullet a \neq z \bullet b \neq z)$	29, EG
31.	$(\exists y)(\exists z)(Sa \bullet Sy \bullet Sz \bullet a \neq y \bullet a \neq z \bullet y \neq z)$	30, EG
32.	$(\exists x)(\exists y)(\exists z)(Sx \bullet Sy \bullet Sz \bullet x \neq y \bullet x \neq z \bullet y \neq z)$	31, EG

QED

10.

1. $Er \bullet \sim Pr \bullet (\forall x)[(Ex \bullet x \neq r) \supset Px]$		
2. $Ej \bullet Pj \bullet (\forall x)[(Ex \bullet Px) \supset x=j]$	/ $(\exists x)(\exists y)\{(Ex \bullet Ey \bullet x \neq y) \bullet (\forall z)[Ez \supset (z=x \vee z=y)]\}$	
3. Er		1, simp
4. Ej		2, simp
| 5. $\sim(\forall z)[Ez \supset (z=r \vee z=j)]$		AIP
| 6. $(\exists z)\sim[Ez \supset (z=r \vee z=j)]$		5, QE
| 7. $\sim[Ea \supset (a=r \vee a=j)]$		6, EI
| 8. $\sim[\sim Ea \vee a=r \vee a=j]$		7, impl
| 9. $\sim\sim Ea \bullet a \neq r \bullet a \neq j$		8, DM
| 10. $Ea \bullet a \neq r \bullet a \neq j$		9, DN
| 11. $(\forall x)[(Ex \bullet x \neq r) \supset Px]$	1, simp	
| 12. $(Ea \bullet a \neq r) \supset Pa$		11, UI
| 13. $Ea \bullet a \neq r$		10, simp
| 14. Pa		12, 13, MP
| 15. $(\forall x)[(Ex \bullet Px) \supset x=j]$	2, simp	
| 16. $(Ea \bullet Pa) \supset a=j$		15, UI
| 17. Ea		14, simp
| 18. $Ea \bullet Pa$		14, 17, conj
| 19. $a=j$	16, 18, MP	
| 20. $a \neq j$	10, simp	
| 21. $a=j \bullet a \neq j$		19, 20, Conj
22. $\sim\sim(\forall z)[Ez \supset (z=r \vee z=j)]$		5-21, IP
23. $(\forall z)[Ez \supset (z=r \vee z=j)]$		22, DN
| 24. $r=j$		AIP
| 25. $\sim Pr$	1, Simp	
| 26. $\sim Pj$	25, 24, IDi	
| 27. Pj		2, Simp
| 28. $Pj \bullet \sim Pj$		27, 26, Conj
29. $r \neq j$		24-28, IP
30. $Er \bullet Ej$		3, 4, Conj
31. $Er \bullet Ej \bullet r \neq j$		30, 29, Conj
32. $Er \bullet Ej \bullet r \neq j \bullet (\forall z)[Ez \supset (z=r \vee z=j)]$		31, 23, Conj
33. $(\exists y)\{Er \bullet Ey \bullet r \neq y \bullet (\forall z)[Ez \supset (z=r \vee z=y)]\}$		32, EG
34. $(\exists x)(\exists y)\{(Ex \bullet Ey \bullet x \neq y) \bullet (\forall z)[Ez \supset (z=x \vee z=y)]\}$		33, EG

QED

Exercises 3.13a

1. $Tmf(m)$
2. $Tf(g(m))m$
3. $Tmf(f(m)) \bullet Tmf(g(m))$
4. $(\exists x)[Sxm \bullet Txg(f(m)) \bullet Txg(g(m))]$
5. $(\exists x)[Sxm \bullet (\forall y)(Sym \supset y=x) \bullet Txg(f(m)) \bullet Txg(g(m))]$
6. $(\forall x){\sim}x=f(x)$
7. ${\sim}(\forall x)[Px \supset (\exists y)x=g(y)]$
8. $(\exists x)[Px \bullet (\exists y)Sf(f(y))x]$

9. $Nt \bullet Pt \bullet Ng(t) \bullet Pg(t)$
10. ${\sim}(\forall x)[(Ox \bullet Nx) \supset Px]$
11. $(\forall x)[(Nx \bullet Ox) \supset Of(x)]$
12. $(\forall x)[Nx \supset {\sim}Pf(x)]$
13. $(\forall x)(\forall y)[(Nx \bullet Ex \bullet Ny \bullet Ey) \supset Ef(x, y)]$
14. $(\forall x)(\forall y)[Nx \supset {\sim}Pf(x,g(x))]$
15. $(\forall x)(\forall y)[(Nx \bullet Ex \bullet Ny \bullet Oy) \supset Ef(x, y)]$
16. $(\forall x)\{Nx \supset [f(x,x) \supset f(x)]\}$

Exercises 3.13b

1.	1. $(\forall x)(Ax \supset Af(x))$		
	2. Aa		
	3. $f(a)=b$	/ Ab	
	4. $Aa \supset Af(a)$		1, UI
	5. $Af(a)$		4, 2, MP
	6. Ab		5, 3, IDi

QED

2.	1. $(\forall x)Bx \equiv Bg(x)$		
	2. $(\forall x)g(x)=f(x,x)$		
	3. Ba		/ $Bf(a,a)$
	4. $Ba \equiv Bg(a)$		1, UI
	5. $[Ba \supset Bg(a)] \bullet [Bg(a) \supset Ba]$		4, Equiv
	6. $Ba \supset Bg(a)$		5, Simp
	7. $Bg(a)$		6, 3, MP
	8. $g(a)=f(a,a)$		2, UI
	9. $Bf(a,a)$		7, 8, IDi

QED

3.	1. $(\forall x)Hf(x)$		
	2. $a=f(b) \bullet b=f(c)$		
	3. $(\forall x)(Hx \supset {\sim}Ix)$	/ $a=f(f(c)) \bullet {\sim}Ia$	
	4. $a=f(b)$	2, Simp	
	5. $b=f(c)$	2, Simp	
	6. $a=f(f(c))$		4, 5, IDi
	7. $Hf(b)$		1, UI
	8. $f(b)=a$	4, IDs	
	9. Ha		7, 8, IDi
	10. $Ha \supset {\sim}Ia$		3, UI
	11. ${\sim}Ia$		10, 9, MP
	12. $a=f(f(c)) \bullet {\sim}Ia$		6, 11, Conj

QED

4.	1. (∀x)[(Bf(x) ⊃ (Cx • Df(f(x)))]	
	2. (∃x)Bf(f(x))	
	3. (∃x)Cf(x) ⊃ (∀x)Ex	/ (∃x)[Df(f(f(x))) • Ef(f(f(x)))]
	4. Bf(f(a))	2, EI
	5. B(f(f(a)) ⊃ [Cf(a) • Df(f(f(a)))]	1, UI
	6. Cf(a) • Df(f(f(a)))	5, 4, MP
	7. Cf(a)	6, Simp
	8. (∃x)Cf(x)	7, EG
	9. (∀x)Ex	3, 8, MP
	10. Ef(f(f(a)))	9, UI
	11. Df(f(f(a)))	6, Simp
	12. Df(f(f(a))) • Ef(f(f(a)))	11, 10, Conj
QED		

5.	1. (∀x)(∀y)[(Fx • Fy) ⊃ Gf(x,y)]	
	2. (∀x)(∀y)[Gf(x,y) ≡ Gf(x,x)]	
	3. (∀x)[Gx ⊃ Gf(x)]	
	4. Fa • Fb	/ Gf(f(a,a))
	5. (∀y)(Fa • Fy) ⊃ Gf(a,y)	1, UI
	6. (Fa • Fb) ⊃ Gf(a,b)	5, UI
	7. Gf(a,b)	6, 4, MP
	8. (∀y)[Gf(a,y) ≡ Gf(a,a)]	2, UI
	9. Gf(a,b) ≡ Gf(a,a)	8, UI
	10. [Gf(a,b) ⊃ Gf(a,a)] • [Gf(a,a) ⊃ Gf(a,b)]	9, Equiv
	11. Gf(a,b) ⊃ Gf(a,a)	10, Simp
	12. Gf(a,a)	11, 7, MP
	13. Gf(a,a) ⊃ Gf(f(a,a))	3, UI
	14. Gf(f(a,a))	13, 12, MP
QED		

6.	1. f(a,b,c)=d	
	2. (∀x)(∀y)(∀z)(∀w){f(x,y,z)=w ⊃ [Jw ∨ Jf(w)]}	
	3. (∀x)(Jx ⊃ Kx)	/ Kd ∨ Kf(d)
	4. (∀y)(∀z)(∀w){f(a,y,z)=w ⊃ [Jw ∨ Jf(w)]	2, UI
	5. (∀z)(∀w){f(a,b,z)=w ⊃ [Jw ∨ Jf(w)]	4, UI
	6. (∀w){f(a,b,c)=w ⊃ [Jw ∨ Jf(w)]	5, UI
	7. f(a,b,c)=d ⊃ [Jd ∨ Jf(d)]	6, UI
	8. Jd ∨ Jf(d)	7, 1, MP
	9. Jd ⊃ Kd	3, UI
	10. Jf(d) ⊃ Kf(d)	3, UI
	11. (Jd ⊃ Kd) • (Jf(d) ⊃ Kf(d))	9, 10, Conj
	12. Kd ∨ Kf(d)	11, 8, CD
QED		

7.

1. (∀x)[(Px • Qx) ⊃ Rf(x)]	
2. (∀x)[Rx ⊃ (∃y)Pxy]	
3. ~(∀x)(Px ⊃ ~Qx)	/ (∃x)(∃y)Pxy
4. (∃x)~(Px ⊃ ~Qx)	3, QE
5. (∃x)~(~Px ∨ ~Qx)	4, Impl
6. (∃x)(~~Px • ~~Qx)	5, DM
7. (∃x)(Px • Qx)	6, DN
8. Pa • Qa	7, EI
9. (Pa • Qa) ⊃ Rf(a)	1, UI
10. Rf(a)	9, 8, MP
11. Rf(a) ⊃ (∃y)Pf(a)y	2, UI
12. (∃y)Pf(a)y	11, 10, MP
13. (∃x)(∃y)Pxy	12, EG

QED

8.

1. (∀x)(∀y)[(Pxy • Qxy) ⊃ ~f(x)=y]	
2. (∀x)(∀y)[Qxy ≡ Qxf(y)]	
3. f(a)=b • f(b)=a	
4. Pab	/ ~Qaa
5. f(a)=b	3, Simp
6. (∀y)[(Pay • Qay) ⊃ ~f(a)=y]	1, UI
7. (Pab • Qab) ⊃ ~f(a)=b	6, UI
8. ~~f(a)=b	5, DN
9. ~(Pab • Qab)	7, 8, MT
10. ~Pab ∨ ~Qab	9, DM
11. ~~Pab	4, DN
12. ~Qab	10, 11, DS
13. (∀y)[Qay ≡ Qaf(y)]	2, UI
14. Qab ≡ Qaf(b)	13, UI
15. [Qab ⊃ Qaf(b)] • [Qaf(b) ⊃ Qab]	14, Equiv
16. Qaf(b) ⊃ Qab	15, Simp
17. ~Qaf(b)	16, 12, MT
18. f(b)=a	3, Simp
19. ~Qaa	17, 18, IDi

QED

9.

1. (∀x)(∀y){Qf(x,y) ⊃ [(Px • Qy) ∨ (Py • Qx)]}	
2. (∀x)[Px ⊃ Qf(x)]	
3. (∀x)Qf(x,f(x))	
4. ~Pa	/ Qa • Pf(a)
5. Qf(a,f(a))	3, UI
6. (∀y){Qf(a,y) ⊃ [(Pa • Qy) ∨ (Py • Qa)]}	1, UI
7. Qf(a,f(a)) ⊃ [(Pa • Qf(a)) ∨ (Pf(a) • Qa)]	6, UI
8. (Pa • Qf(a)) ∨ (Pf(a) • Qa)	7, 5, MP
9. ~Pa ∨ ~Qf(a)	4, Add
10. ~(Pa • Qf(a))	9, DM
11. Pf(a) • Qa	8, 10, DS
12. Qa • Pf(a)	11, Com

QED

10.

1. (∀x)(∀y){Pf(x,y) ⊃ [(Px • Py) ∨ (Qx • Qy)]}	
2. (∀x)[Px ⊃ Pf(f(x)]	
3. (∀x)Pf(x, f(f(x)))	
4. (∃x)~Qx	/ (∃x)Pf(f(x))
5. ~Qa	4, EI
6. Pf(a,f(f(a)))	3, UI
7. (∀y){Pf(a,y) ⊃ [(Pa • Py) ∨ (Qa • Qy)]}	1, UI
8. Pf(a,f(f(a))) ⊃ [(Pa • Pf(f(a))) ∨ (Qa • Qf(f(a)))]	7, UI
9. (Pa • Pf(f(a))) ∨ (Qa • Qf(f(a)))	8, 6, MP
10. ~Qa ∨ ~Qf(f(a))	5, Add
11. ~(Qa • Qf(f(a)))	10, DM
12. (Qa • Qf(f(a))) ∨ (Pa • Pf(f(a)))	9, Com
13. Pa • Pf(f(a))	12, 11, DS
14. Pf(f(a))	13, Simp
15. (∃x)Pf(f(x))	14, EG

QED

Exercises 3.14

1. (∃X)Xl • (∃X)~Xl
2. (∀X)(Xc ≡ ~Xd)
3. (∃X)(∃Y)[Xr • Yr • (∃x)~(Xx≡Yx)]
4. (∀x)[Px ⊃ (∃X)(Xt • Xx)]
5. (∀x)[Px ⊃ (∃X)(∃y)(My • Xx • Xy)]
6. (∃x)[Cx • (∃X)(Xx • Xe)]
7. (∃x)[(Fx • Sx) • (∃X)(Xg • Xx)]
8. (∀x){Px ⊃ (∀y)[By ⊃ (∃X)(Xx • Xy)]}
9. (∃X)(Xa • Xf(a))
10. (∀X)(Xg(r) ⊃ Xr)
11. (∃X)(∀x)~Xx
12. (∃X){(∀x)(∀y)(∀z)[(Xxy • Xyz) ⊃ Xxz]}
13. (∃z)(∀X)[(∀x)(∀y)(Xxy ≡ Xyx) ⊃ (∀w)(~Xzw • ~Xwz)]
14. (∃X)[(∀x)Xxx • (∀x)(∀y)(Xxy ≡ Xyx)]
15. (∃w)(∀X){(∀x)(∀y)(∀z)[(Xxy • Xyz) ⊃ Xxz] ⊃ (∀v)(~Xwv • ~Xvw)}
16. (∀x)(∀y){[Sx • Sy • ~(∀X)(Xx ≡ Xy)] ⊃ ~(∃X)(Xx • Xy)}
17. (∀x)(∀y)[x≠y ⊃ (∃X)(Xx ≡ ~Xy)]
18. (∀x)(∀y){[Sx • Ux • Sy • Uy • (∃X)(Xx • Xy)] ⊃ x=y}
19. (∀x)(∀y)[~(∃X)(Xx • Xy) ⊃ (~x=f(y) • ~y=f(x))]
20. (∀x)(∀y)[~(∃X)(Xx • Xy) ⊃ (~Uxy • ~Uyx)]